Varying-alpha cosmologies with potentials

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We generalize the Bekenstein-Sandvik-Barrow-Magueijo model for the variation of the fine-structure "constant," α, to include an exponential or inverse power-law self-potential for the scalar field φ which drives the time variation of α, and consider the dynamics of φ in such models. We find solutions for the evolution of φ or α in matter-, radiation-, and dark-energy-dominated cosmic eras. In general, the evolution of φ is well determined solely by either the self-potential or the coupling to matter, depending on the model parameters. The results are general and applicable to other models where the evolution of a scalar field is governed by a matter coupling and a self-potential. We find that the existing astronomical data stringently constrains the possible evolution of α between redshifts $z \simeq 1$–3.5 and the present, and this leads to a very strong limit on the allowed deviation of the potential from that of a pure cosmological constant.

I. INTRODUCTION

For the first time there is a body of detailed astronomical evidence consistent with the time variation of a traditional constant of nature. The observational programme of Webb *et al.* [1,2] has completed detailed analyses of 128 Keck-HIRES quasar absorption line systems at redshifts $0.5 < z < 3$ using the many-multiplet method to compare separations between line separations affected by special-relativistic effects and found evidence consistent with the fine-structure constant at redshift z, $\alpha(z)$, having been *smaller* in the past, at redshifts $z \simeq 1$–3.5. The shift in the value of α between its value $\alpha(z)$ at redshift z and its present-day value $\alpha(0)$, for all the data is given provisionally by

$$\Delta\alpha/\alpha \equiv [\alpha(z) - \alpha(0)]/\alpha(0) = (-0.57 \pm 0.10) \times 10^{-5}.$$

Subsequent reduction of another data set of 23 Ultraviolet and Visual Echelle Spectrograph on the Very Large Telescope quasar absorption systems $0.4 \leq z \leq 2.3$ by Chand *et al.*, [3,4] using a partial version of the many-multiplet method at first produced a result consistent with no variation in α, with an unusually small uncertainty, $\Delta\alpha/\alpha = (-0.06 \pm 0.06) \times 10^{-5}$. However, the data reduction did not allow $\Delta\alpha$ to be a free parameter in the data fitting, and a reanalysis of the same data set by Murphy *et al.* [5] using the full many-multiplet method increases the uncertainties sixfold, and leads to a revised bound of

$$\Delta\alpha/\alpha = (-0.64 \pm 0.36) \times 10^{-5}.$$

Any present-day variation of α can also be constrained by direct laboratory comparisons of clocks based on different atomic frequency standards over a period of months or years. Until recently, the most stringent atomic clock constraints on any current temporal variation of α were

$$\dot{\alpha}/\alpha = (-3.3 \pm 3.0) \times 10^{-16} \text{ yr}^{-1},$$

which arose by combining measurements of the frequencies of Sr [6], Hg [7], Yb [8], and H [9] relative to Cs; Cingöz *et al.* [10] have also recently reported a less stringent limit of $\dot{\alpha}/\alpha = -(2.7 \pm 2.6) \times 10^{-15} \text{ yr}^{-1}$. If the systematic errors can be fully understood, an ultimate sensitivity of 10^{-18} yr^{-1} may be possible with this method [11]. If a linear variation in α is assumed then the Murphy *et al.* quasar measurements equate to $\dot{\alpha}/\alpha = (6.4 \pm 1.4) \times 10^{-16} \text{ yr}^{-1}$ [1,2]. If the variation is due to a light scalar field described by a theory like that of Bekenstein and Sandvik, Barrow and Magueijo (BSBM) [12,13], then the rate of change in the constants is exponentially damped during the recent dark-energy-dominated era of accelerated expansion, and one typically predicts a present-day value of

$$\dot{\alpha}/\alpha = 1.1 \pm 0.3 \times 10^{-16} \text{ yr}^{-1}$$

by direct extrapolation from the Murphy *et al.* data [1,2]. This is not ruled out by the atomic clock constraints mentioned above. For comparison, the Oklo natural reactor constraints, which are based on the need for the $\text{Sm}^{149} + n \rightarrow \text{Sm}^{147} + \gamma$ neutron capture resonance at 97.3 MeV to have been present 1.8–2 Gyr ago at $z = 0.15$, as first pointed out by Shlyakhter [14], are currently [15] $\Delta\alpha/\alpha = (-0.8 \pm 1.0) \times 10^{-8}$ or $(8.8 \pm 0.7) \times 10^{-8}$ (because of the double-valued character of the neutron capture cross section with reactor temperature) and [16] $\Delta\alpha/\alpha > 4.5 \times 10^{-8}$ (6σ), when the nonthermal neutron spectrum is taken into account. However, there remain significant environmental uncertainties regarding the reactor's early history and the relationship between changes in the resonance energy level and those in the values of any underlying constants. For reviews of the wider issue of varying con-

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stants in addition to α, see the reviews in Refs. [17], and for some implications of the unification of fundamental forces see Refs. [18].

Recently, Rosenband *et al.* [19] measured the ratio of aluminum and mercury single-ion optical clock frequencies, f_{Al+}/f_{Hg+}, at intervals over a period of about a year. From these measurements, the linear rate of change in this ratio was found to be $(-5.3 \pm 7.9) \times 10^{-17}$ yr^{-1} (but see Ref. [20] for some refinements). These measurements provide the strongest limit yet on any temporal drift in the value of α

$$\dot{\alpha}/\alpha = (-1.6 \pm 2.3) \times 10^{-17} \text{ yr}^{-1}.$$

This limit is strong enough to exclude theoretical explanations of the change in α reported by Webb *et al.* [1,2] based on the slow variation of an effectively massless scalar field [13], even allowing for the damping by cosmological acceleration, unless there is a significant effect that slows the locally observed effects of changing α on cosmological scales (for a detailed analysis of global-local coupling of variations in constants (see Refs. [21]).

Theories in which α varies will in general lead to violations of the weak equivalence principle. This is because the α variation is carried by a scalar field, φ, and this couples differently to different nuclei because they contain different numbers of electrically charged particles (protons). The theory discussed here has the interesting consequence of leading to a relative acceleration of order 10^{-13} [22] if the free coupling parameter is fixed to the value given in Eq. (6) using a best fit of the theory's cosmological model to the quasar observations of Refs. [1,2]. Other predictions of weak equivalence principle violations have also been made in Refs. [23–25]. The observational upper bound on this parameter from direct experiment is just an order of magnitude larger, at 10^{-12}, and limits from the motion of the Moon are of similar order, [26], but space-based tests planned for the STEP mission are expected to achieve a sensitivity of order 10^{-18} and will provide a completely independent check on theories of time-varying e and α [27,28].

In view of this tension between direct local measurements and astronomical measurements of the fine-structure "constant" it is important to explore the widest possible range of self-consistent theoretical models for the time evolution of α so as to understand the possible evolutions of $\Delta\alpha/\alpha$ over the range $0 < z < 6$ that spans the astronomical, geochemical, and laboratory measurements. In the remainder of this paper we will present cosmological extensions to the simple BSBM scalar-field models for varying α that include a nonzero self-interaction potential, $V(\varphi)$ for the scalar field, φ, carrying the spacetime evolution of α. We will consider two representative theories, where V has exponential and power-law variation, respectively, and determine the solutions for the cosmological

evolution and the time variation of α during the radiation, dust, and dark-energy dominated eras of the Universe.

The organization of this paper is as follows: in Secs. II and III we present the theory of varying α based on the coupling of a scalar field to the electromagnetically charged matter and list the relativistic equations for the investigations of this theory. In Secs. IV and V we review, respectively, the cosmological evolutions of the scalar field φ for the model with no scalar-field self-interaction potential (just coupling with matter) and for quintessence models with exponential and inverse power-law self-potentials. Section VI is devoted to an investigation of how the scalar field φ evolves if both the matter coupling term and the bare self-potential are nonzero, which is supplemented by the numerical examples shown in Sec. VII. Finally, conclusions are drawn in Sec. VIII.

II. BSBM SCALAR-FIELD THEORIES FOR VARYING α

There are a number of possible theories allowing for the variation of the fine-structure constant, α. In the simplest cases we take c and $\hbar$ to be constants and attribute variations in α to changes in e or the permittivity of free space (see [29] for a discussion of the meaning of this choice). This is done by letting e take on the value of a real scalar field which varies in space and time. Thus $e_0 \rightarrow e = e_0 \epsilon(x^\mu)$, where ϵ is a dimensionless scalar field and e_0 is a constant denoting the present value of e. This operation implies that some well established assumptions, like charge conservation, must give way [30]. Nevertheless, the principles of local gauge invariance and causality are maintained, as is the scale invariance of the ϵ field (under a suitable choice of dynamics) and there is no conflict with local Lorentz invariance or covariance.

The dynamics are then constructed as follows. Since e is the electromagnetic coupling, the ϵ field couples to the gauge field as ϵA_μ in the Lagrangian and the gauge transformation which leaves the action invariant is $\epsilon A_\mu \rightarrow \epsilon A_\mu + \chi_{,\mu}$, rather than the usual $A_\mu \rightarrow A_\mu + \chi_{,\mu}$. The gauge-invariant electromagnetic field tensor is therefore

$$F_{\mu\nu} = \frac{1}{\epsilon}((\epsilon A_\nu)_{,\mu} - (\epsilon A_\mu)_{,\nu}), \tag{1}$$

which reduces to the usual form when ϵ is constant. The electromagnetic part of the action is still

$$S_{em} = -\int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \tag{2}$$

and the dynamics of the ϵ field are controlled by the kinetic term

$$S_\epsilon = -\frac{1}{2} \frac{\hbar}{l^2} \int d^4x \sqrt{-g} \frac{\epsilon_{,\mu} \epsilon^{,\mu}}{\epsilon^2}, \tag{3}$$

as in dilaton theories. Here, l is the characteristic length scale of the theory, introduced for dimensional reasons.

This constant length scale gives the scale down to which the electric field around a point charge is accurately Coulombic. The corresponding energy scale, $\hbar c/l$, has to lie between a few tens of MeV and the Planck scale, $\sim 10^{19}$ GeV to avoid conflict with the experiment. This generalization of the scalar theory proposed by Bekenstein [12] was made by Sandvik, Magueijo, and Barrow [31–34] and will be referred to as the BSBM theory. It includes the gravitational effects of φ and gives the field equations

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\varphi} + T_{\mu\nu}^{\text{em}} e^{-2\varphi}). \quad (4)$$

The stress tensor of the φ field is derived from the Lagrangian $\mathcal{L}_{\varphi} = -\frac{\omega}{2} \partial_{\mu}\varphi\partial^{\mu}\varphi$ and the φ field obeys the equation of motion

$$\Box\varphi = \frac{2}{\omega} e^{-2\varphi} \mathcal{L}_{\text{em}} \quad (5)$$

where we have defined the coupling constant $\omega = (c)/l^2$. This constant is of order ~ 1 if, as in [13], the energy scale is similar to the Planck scale. It is clear that $\mathcal{L}_{\text{em}}$ vanishes for a sea of pure radiation since then $\mathcal{L}_{\text{em}} = (E^2 - B^2)/2 = 0$. We therefore expect the variation in α to be driven by electrostatic and magnetostatic energy components rather than electromagnetic radiation and with $\hbar = c = 1$, the fine-structure constant is given by

$$\alpha/\alpha_0 \equiv e^2/e_0^2 = \exp(2\varphi).$$

The considerations raised by Duff [35] do not impact upon well-defined varying constant theories like this, even if they appear dimensionful. The presence of a new field, like φ, always requires a second-order energy conservation equation, like Eq. (5) and the integration of this equation always leads to a new integration constant, φ_0, with the same dimensions as φ and so the evolution of the dimensionless quantity φ/φ_0 involves no ambiguities under redefinitions of units.

In order to make quantitative predictions we need to know how much of the nonrelativistic matter contributes to the right-hand side of Eq. (5). This is parametrized by $\zeta \equiv \mathcal{L}_{\text{em}}/\rho$, where ρ is the energy density, and for baryonic matter $\mathcal{L}_{\text{em}} = E^2/2$. For protons and neutrons ζ_p and ζ_n can be estimated from the electromagnetic corrections to the nucleon mass, 0.63 MeV and -0.13 MeV, respectively [24]. This correction contains the $E^2/2$ contribution (always positive), but also terms of the form $j_{\mu}a^{\mu}$ (where j_{μ} is the quarks' current) and so cannot be used directly. Hence, we take a representative value $\zeta_p \approx \zeta_n \sim 10^{-4}$. Furthermore, the cosmological value of ζ (denoted ζ_m) has to be weighted by the fraction of matter that is nonbaryonic. Hence, ζ_m depends strongly on the nature of the dark matter and can take both positive and negative values depending on which of Coulomb energy or magnetostatic energy dominates the dark matter of the Universe. It could be that $\zeta_{\text{DM}} \approx -1$ (superconducting cosmic strings, for which $\mathcal{L}_{\text{em}} \approx -B^2/2$), or $\zeta_{\text{DM}} \ll 1$ (neutrinos) where a

subscript DM means dark matter. BBN predicts an approximate value for the baryon density of $\Omega_B \approx 0.03$ (where Ω_B is the density of matter in units of the critical density $3H^2/8\pi G$) with a Hubble parameter of $H = 60$ Km s^{-1} Mpc^{-1}, implying $\Omega_{\text{DM}} \approx 0.3$. Thus, depending on the nature of the dark matter, ζ_m can be virtually anything between -1 and $+1$. The uncertainties in the underlying quark physics and especially the constituents of the dark matter make it difficult to impose more certain bounds on ζ_m.

There are a number of conclusions that can be drawn from the study of the simple BSBM models with $\zeta_m < 0$. These models gave a good fit to the varying α implied by the quasar data of Refs. [1,2]. There is just a single parameter to fit and this is given by the choice [13]

$$-\frac{\zeta_m}{\omega} = (2 \pm 1) \times 10^{-4}. \quad (6)$$

The simple solutions of the BSBM theory predict a slow (logarithmic) time increase of α during the dust era of $k = 0$ Friedmann universes. The cosmological constant turns off the time variation of α at the redshift when the Universe begins to accelerate ($z \sim 0.7$) and so there is no conflict between the α variation seen in quasars at $z \sim 1$–3.5 and the limits on possible variation of α deduced from the operation of the Oklo natural reactor [14,15] (even assuming that the cosmological variation applies unchanged to the terrestrial environment). The reactor operated 1.8 billion years ago at a redshift of only $z \sim 0.1$ when no significant variations were occurring in α. The slow logarithmic increase in α also means that we would not expect to have seen any effect yet in the anisotropy of the microwave backgrounds [36–38]: the value of α at the last scattering redshift, $z = 1000$, is only 0.005% lower than its value today. Similarly, the essentially constant evolution of α predicted during the radiation era leads us to expect no measurable effects on the products of big bang nucleosynthesis (BBN) [39] because α was only 0.007% smaller at BBN than it is today. This does not rule out the possibility that unification effects in a more general theory might require variations in weak and strong couplings, or their contributions to the neutron-proton mass difference, which might produce observable differences in light-element nucleosynthesis, and new constraints on varying α, at $z \sim 10^9$–10^{10}. By contrast, varying-alpha cosmologies with $\zeta > 0$ lead to bad consequences unless the scalar field driving the alpha variations is a "ghost" field, with negatively coupled kinetic energy, in which case there can be interesting cosmological consequences [40]. The fine-structure constant falls rapidly at late times and the variation is such that it comes to dominate the Friedmann equation for the cosmological dynamics. We regard this as a signal that such models are astrophysically ruled out and perhaps are also mathematically badly behaved.

The earlier analyses of the cosmological solutions of the BSBM equations considered the situation in which the scalar-field driving variations in α has no self-interaction potential, $V(\varphi) \equiv 0$. In this paper, we are going to explore some of the consequences for the time variation of $\alpha \equiv \exp(2\varphi)$ that arise when we introduce a nonzero potential for the scalar-field driving the variations in α. We will consider the representative cases of the exponential potential $V(\varphi) = V_0 \exp(-\lambda\varphi)$ and inverse power-law potential $V(\varphi) = V_* \varphi^{-\gamma}$, and classify the new behaviors that arise for $\lambda, \gamma \neq 0$. We note that the cases of $\lambda, \gamma = 0$ correspond to $V = V_0$, which is equivalent to the presence of a cosmological constant. The solutions for such scenarios were found in our earlier studies and φ relaxes quickly to a constant asymptotic value once the expansion starts to accelerate.

III. COSMOLOGICAL EQUATIONS

In the BSBM theory, the total action of the Universe is given by

$$S = \int d^4x \sqrt{-g}(\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_\varphi + \mathcal{L}_{\text{em}}e^{-2\varphi}).$$

The Universe is described by a homogeneous and isotropic Friedmann metric with expansion scale factor $a(t)$. The Friedmann equation is given by

$$H^2 = \tfrac{1}{3}[\rho_m(1 + |\zeta|e^{-2\varphi}) + \rho_r e^{-2\varphi} + \rho_\varphi + \rho_\Lambda], \quad (7)$$

where we assume that the Universe is spatially flat; the quantities ρ_m, ρ_r, ρ_φ, ρ_Λ are the energy densities in nonrelativistic matter, relativistic matter, scalar field, and cosmological constant (so ρ_Λ is a constant), respectively. We will first consider the case where the scalar field φ has no potential term, and then consider the cases with exponential and power-law potentials.

The conservation equations for matter and radiation are given as

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (8)$$

$$\dot{\rho}_r + 4H\rho_r = 2\rho_r\dot{\varphi}, \quad (9)$$

and the scalar-field equation of motion is

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V(\varphi)}{\partial \varphi} = \frac{2|\zeta|}{\omega}\rho_m e^{-2\varphi}. \quad (10)$$

Taking the time derivative of Eq. (7), and using Eqs. (8)–(10), we get

$$\dot{H} = -\tfrac{1}{2}[\rho_m(1 + |\zeta|e^{-2\varphi}) + \tfrac{4}{3}\rho_r e^{-2\varphi} + \omega\dot{\varphi}^2]. \quad (11)$$

In a universe filled with nonrelativistic matter and the scalar field, $\rho_\Lambda = \rho_r = 0$ this equation reduces to

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right].$$

IV. THE CASE OF $\zeta < 0$: V CONSTANT

This was the situation analyzed in the original presentation of the BSBM theory in Refs. [31–34] and extended to include higher-order corrections in Ref. [41], small perturbations [42], and a linearized potential in Ref. [43]. The structure of the cosmological solutions has an expected feature. The cosmological dynamics of the scale factor $a(t)$, controlled by the Friedmann equation, is not influenced to leading order by the small variations in φ. However, the cosmological variation of $a(t)$ has a significant effect on the dynamics of φ, and hence upon the evolution of the fine-structure constant $\alpha(t)$. The key results for a cosmological model that evolves through a radiation-CDM-vacuum energy-dominated sequence of three phases are as follows:

During the radiation era in which $a = t^{1/2}$, there is an exact solution of

$$(\dot{\varphi}a^3)\dot{} = N\exp(-2\varphi)$$

where $N > 0$ is a constant defined by

$$N \equiv -\frac{2\mathsf{s}}{\omega}\rho_m a^3$$

given by

$$\varphi = \tfrac{1}{2}\log(8N) + \tfrac{1}{4}\log(t).$$

For physically realistic choices of the parameters the logarithmic term is never significant during the radiation era of our Universe and φ is constant then, which would be expected since $E^2 = B^2$ for the radiation equilibrium which means that $\zeta \sim (E^2 - B^2)/(E^2 + B^2)$ is effectively zero and φ constant.

During the matter-dominated era $a = t^{2/3}$, and there is a late-time asymptotic series solution of the form

$$\varphi \sim \frac{1}{2}\ln[2N\log(t/t_0)] + Ct^{-1} - \frac{1}{2}\sum_{n=1}^\infty \frac{(n-1)!}{[\log(t/t_0)]^n},$$

$$\varphi \to \frac{1}{2}\log[2N\log(t)],$$

with C and t_0 constants, so $\alpha \propto \exp(2\varphi)$ grows slowly, as $\log(t)$.

During a late-time era dominated by a constant vacuum energy density, $\rho_\Lambda = 3H_0^2$, with de Sitter expansion of the form $a = \exp(H_0 t)$ we have late-time solutions of the form

$$\varphi \sim \varphi_0 + B\exp(-3H_0 t) - \frac{N(3H_0 t + 1)}{9H_0^2}$$

$$\times \exp(-2C - 3H_0 t), \qquad \varphi \to \varphi_0, \quad (12)$$

where φ_0, B, and C are constants. This case corresponds to the addition of a constant potential $V = V_0$ for the scalar field and we see that the effect is to turn off all time variations in φ, and hence α. A constant asymptote for $\varphi(t)$ also occurs for any accelerated expansion in which

$a = t^n$, with $n \geq 1$. Any potential of the form

$$V(\varphi) = V_0 + U(\varphi)$$

where U falls off as $\exp(-\mu\varphi^n)$ for large φ, with $n > 1$, will result in φ approaching a constant value as $t \to \infty$ so long as the kinetic energy of the field is negligible compared to the matter density, which is the physically realistic situation. In contrast, the kinetic energy $\frac{1}{2}\dot{\varphi}^2$ may dominate as $t \to 0$. If it does then it leads to evolution of the scale factor with $a = t^{1/3}$ and an exact solution for the scalar-field evolution of the form [41]

$$\varphi = \frac{1}{2} \log\left(\frac{N}{4}\right) - \log(E) + \frac{1}{2}\log(t)$$
$$+ \log\left[\left(\frac{t_0}{t}\right)^E + \left(\frac{t}{t_0}\right)^E\right],$$

where E and t_0 are constants; this solution approaches $\varphi = (\frac{1}{2} - E)\log(t)$ as $t \to 0$ and $\varphi = (\frac{1}{2} + E)\log(t)$ as $t \to \infty$, so the fine-structure constant evolves as $\alpha \propto t^{1\pm2E}$ in these limits if the kinetic energy dominates.

V. THE CASE OF $\zeta = 0$: α CONSTANT

When $\zeta = 0$, there is no coupling of the scalar field to the electromagnetic matter, the problem reduces to the cosmology of a scalar field in the presence of a perfect fluid and there is no variation of the fine-structure constant α. In order to include the effects of a self-interaction potential, we shall assume two popular choices, an exponential potential and an inverse power-law potential for the scalar field,

$$V(\varphi) = V_0 \exp(-\lambda\varphi), \qquad V(\varphi) = V_*\varphi^{-\gamma}$$

in which λ and γ are dimensionless constants and $V_0 \geq 0$, $V_* \geq 0$ are constants with dimensions $[\text{mass}]^4$ and $[\text{mass}]^{4+\gamma}$, respectively. These two potentials have been studied extensively; they can be used to obtain power-law inflation when the scalar field is the only matter source, and have scaling solutions where the energy density of the scalar field evolves in proportion to the density of the dominant fluid component of the Universe in the presence of matter and radiation. With the potential added, the field equations become

$$3H^2 = \rho_m + \tilde{\rho}_r + \frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \rho_\Lambda, \quad (13)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad (14)$$

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (15)$$

$$\dot{\tilde{\rho}}_r + 4H\tilde{\rho}_r = 0, \quad (16)$$

in which we have defined $\tilde{\rho}_r \equiv \rho_r \exp(-2\varphi)$. We shall discuss two particular potentials in turn below in prepara-

tion for the discussion of the situation where varying α is introduced.

A. Exponential potential

For the exponential potential it is well-known [44,45] that scaling solutions for φ exist when the Universe is dominated by either radiation or matter. We summarize these solutions here and also derive a leading-order solution for the scalar field in a universe dominated by dark energy (the dark energy is not due to the scalar field φ here, but to the other matter).

1. Radiation-dominated solution

In the radiation-dominated era, we could neglect the nonrelativistic matter species and vacuum energy density ($\rho_m = \rho_\Lambda = 0$) and obtain the following solution:

$$a \propto t^{1/2}, \quad (17)$$

$$H = \frac{1}{2t}, \quad (18)$$

$$\tilde{\rho}_r = \tilde{\rho}_{r0}\frac{t_0^2}{t^2}, \quad (19)$$

$$\varphi = \varphi_0 + \frac{2}{\lambda}\log\frac{t}{t_0}, \quad (20)$$

where $\tilde{\rho}_{r0}$, t_0, and φ_0 are constants. It is easy to see that the scalar-field energy density is given by

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \equiv \left(\frac{2}{\lambda^2} + \tilde{V}_0 t_0^2\right)\frac{1}{t^2} \quad (21)$$

in which we have defined $\tilde{V}_0 = V_0 \exp(-\lambda\varphi_0)$. Thus, $\rho_\varphi \propto t^{-2}$ scales in proportion to the radiation energy density $\tilde{\rho}_r$ and their ratio is kept constant during the evolution.

Note the Friedmann equation and the scalar-field equation of motion give two algebraic relations between the constants defining the scaling solution,

$$\frac{3}{4} = \left[\tilde{\rho}_{r0}t_0^2 + \frac{2}{\lambda^2} + \tilde{V}_0 t_0^2\right] \quad \text{and} \quad 1 = \lambda^2\tilde{V}_0 t_0^2,$$

and so

$$\tilde{V}_0 t_0^2 = \frac{1}{\lambda^2} \quad \text{and} \quad \tilde{\rho}_{r0}t_0^2 = \frac{3(\lambda^2 - 4)}{4\lambda^2}.$$

Hence, the constant fractional energy densities are given by

$$\Omega_r = \frac{\lambda^2 - 4}{\lambda^2}, \quad (22)$$

$$\Omega_\varphi = \frac{4}{\lambda^2}. \quad (23)$$

2. Matter-dominated solution

Similarly, in the matter-dominated era we can neglect the radiation and vacuum densities to obtain the following solution:

$$a \propto t^{2/3}, \tag{24}$$

$$H = \frac{2}{3t}, \tag{25}$$

$$\rho_m = \rho_{m0} \frac{t_0^2}{t^2}, \tag{26}$$

$$\varphi = \varphi_0 + \frac{2}{\lambda} \log\frac{t}{t_0}, \tag{27}$$

where ρ_{m0}, t_0 and φ_0 are new constants. Here, Eq. (21) still holds and ρ_φ now scales as ρ_m. The Friedmann equation and the scalar-field equation of motion give two algebraic relations between these quantities

$$\frac{4}{3} = \left[\rho_{m0}t_0^2 + \frac{2}{\lambda^2} + \tilde{V}_0 t_0^2\right] \quad \text{and} \quad 2 = \lambda^2 \tilde{V}_0 t_0^2,$$

which lead to

$$\tilde{V}_0 t_0^2 = \frac{2}{\lambda^2} \quad \text{and} \quad \rho_{m0} t_0^2 = \frac{4(\lambda^2 - 3)}{3\lambda^2},$$

and so the constant fractional energy densities are given by

$$\Omega_m = \frac{\lambda^2 - 3}{\lambda^2}, \tag{28}$$

$$\Omega_\varphi = \frac{3}{\lambda^2}. \tag{29}$$

Note that in order for $\Omega_\varphi < 1$ during the radiation era, $\lambda^2 > 4$ is required, and this then ensures $\Omega_\varphi < 1$ during the dust era and $\Omega_m > 0$.

3. Dark-energy-dominated solution

Recent observations suggest that the Universe is currently, and will remain, dominated by a gravitationally repulsive form of matter dubbed dark energy. In the simplest scenario, this is just a cosmological constant for which the expansion rate of the Universe will tend to a constant, while in other models it can be exotic matter, or changes to the law of gravitation, which drive a different future evolution of the Universe. Here we consider the evolution of our φ driven by the exponential potential in the background of simple dark-energy domination. For simplicity we consider just two cases, where the cosmic expansion factor is either exponential, $a = \exp(H_0 t)$, H_0 constant, or a power law in time, $a \propto t^\beta$ with $\beta > 1$.

(a) *Case 1*: This is a cosmological constant (Λ) dominated universe, for which the scale factor evolves as

$a \propto \exp(\varsigma t)$ where $\varsigma \equiv \sqrt{\Lambda/3}$ and so the equation of motion for φ becomes

$$\ddot{\varphi} + 3\varsigma\dot{\varphi} = W \exp(-\lambda\varphi) \tag{30}$$

where we have defined $W \equiv \lambda V_0$. This equation is very similar to the scalar field equation of motion for the BSBM model in the dust-dominated era. It has no closed analytical solution and we shall seek a self-consistent approximate solution following the logic in Ref. [31]. We start from the ansatz that in the Λ dominated era the field φ is slowly rolling, and it is easy to obtain the slow-roll solution $\varphi \sim \frac{1}{\lambda} \times \log(\lambda W t/3\varsigma)$ by setting $\ddot{\varphi} = 0$. Let us next make the following approximation by an asymptotic series:

$$\varphi = \frac{1}{\lambda} \log\left[\frac{\lambda W}{3\varsigma} t\right] + \sum_{n=1}^{\infty} a_n t^{-n}$$

where a_n are some constant coefficients. Substituting this back into the scalar field equation of motion Eq. (30) we have

$$-\frac{1}{\lambda t^2} + \sum_{n=1}^{\infty} n(n+1)\frac{a_n}{t^{n+2}} + \frac{3\varsigma}{\lambda t} - \sum_{n=1}^{\infty} n\frac{a_n}{t^{n+1}}$$

$$= \frac{3\varsigma}{\lambda t} \exp\left[-\lambda \sum_{n=1}^{\infty} \frac{a_n}{t^n}\right] \to \frac{3\varsigma}{\lambda t},$$

as $t \to \infty$. Choose appropriate a_n so that the terms $1/t^r$ with $r \geq 2$ cancel, we find that the solution for φ can be written as

$$\varphi = \frac{1}{\lambda} \log\left[\frac{\lambda W}{3\varsigma} t\right] - \frac{1}{2}\left[\frac{1}{t} + \frac{1}{t^2} + \frac{2}{t^3} + \cdots \right.$$

$$\left. + \frac{(r-1)!}{t^r} + \cdots\right]. \tag{31}$$

It is clear that as time grows, the asymptotic series becomes less important and so the slow-roll solution is ever improved. Equation (31) is a good approximation when t is large as we have seen in the derivation; when t is small, Eq. (30) could be linearized as

$$\ddot{\varphi} + 3\varsigma\dot{\varphi} = W$$

where we assume the initial value of φ is zero. The solution is then

$$\varphi = \varphi_c + A \exp(-3\varsigma t) + \frac{Wt}{3\varsigma} \to \varphi_c + \frac{Wt}{3\varsigma} \tag{32}$$

where φ_c and A are constants of integration. The linear term in t seems to be the second Taylor term of the slow-roll solution. We can see that in both solutions the scalar field φ will not tend to constant when t goes large, which is in contrast to the case of BSBM.

(b) *Case 2*: This is described by $a \propto t^n$ ($n > 1$), for which the scalar field equation of motion could be written as

$$\ddot{\varphi} + \frac{3n}{t} \dot{\varphi} = W \exp(-\lambda\varphi) \qquad (33)$$

which has an exact solution

$$\varphi = \frac{1}{\lambda} \log\frac{\lambda W}{2(3n-1)} + \frac{2}{\lambda} \log t. \qquad (34)$$

Again, we find a logarithmic behavior of φ in the acceleration era, which means that φ will never approach a constant. This is, of course, not surprising because we know that the exponential potential has tracking behavior for any power-law background expansion with $n > 1/3$, no matter whether it is $n < 1$ (matter and radiation dominations) or $n > 1$ (dark-energy domination).

Of course, we also need to justify the assumption that the energy density in the scalar-field is always subdominant. In case 1 this is obvious because the Λ has a constant energy density while the scalar field has a p/ρ ratio which is greater than -1, meaning that its energy density decays continuously. For case 2 we have $\rho_\varphi \propto t^{-2} \propto \rho_{\text{DE}}$ and it again exactly tracks the dominant component in the Universe. In both cases there is no way for φ to come to dominate the total energy density.

B. Inverse power-law potential

We turn next to the inverse power-law potential $V = V_* \varphi^{-\gamma}$, with γ a positive constant. It is well-known [46] that this potential also permits tracking behavior of the scalar field φ.

1. Radiation- and matter-dominated solutions

Suppose the background universe expands according to $a \propto t^n$ and the energy density in φ is only subdominant, then φ has the solution

$$\varphi = A t^{2/\gamma+2} \qquad (35)$$

where A is constant to be fixed. To determine the value of A, take the time derivatives of φ

$$\dot{\varphi} = \frac{2A}{\gamma+2} t^{-(\gamma/\gamma+2)}, \qquad \ddot{\varphi} = -\frac{2A\gamma}{(\gamma+2)^2} t^{-(2\gamma+2/\gamma+2)} \qquad (47)$$

and insert them together with $H = n/t$ into the scalar field equation of motion, we get an algebraic equation for A which has the solution

$$A = \left[\frac{\gamma(\gamma+2)^2 V_*}{6n(\gamma+2) - 2\gamma} \right]^{1/\gamma+2}. \qquad (36)$$

Since $\varphi \propto t^{1/\gamma+2}$, it is easy to see that $\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \propto t^{-(2\gamma/\gamma+2)} \to t^{-2}$ for $\gamma \gg 1$. As the domi-

nant component in the Universe also has an energy density scaling as t^{-2}, we see that the energy density of φ simply tracks the dominant component, which is radiation in the radiation era and dust in the matter era.

Note that this tracking behavior is only approximate for $\gamma \gg 1$, while in reality ρ_φ decays slower than ρ_{dominant}. This means that the fractional energy density of the scalar field φ is ever increasing and eventually will no longer be subdominant. However, for enough large γ this will take a very long time so the issue will not bother us for some time.

2. Dark-energy-dominated solution

We next consider the evolution of φ in a dark-energy-dominated universe. Again, we consider two cases, case 1 for Λ domination where H is a constant and case 2 for a power-law inflation $a \propto t^n$ ($n > 1$). Obviously case 2 has the same behavior as in the radiation or matter-dominated universes but with the value of n in Eq. (36) changed, and so we will not consider it again here except stating that in the $a \propto t^n$ dark-energy era the field φ does not stop growing.

In the first case, of Λ domination, we now write the scalar field equation of motion as

$$\ddot{\varphi} + 3\mathsf{s}\dot{\varphi} = \frac{\gamma V_*}{\varphi^{\gamma+1}}. \qquad (37)$$

The slow-roll solution to this equation is

$$\varphi \sim \left[\frac{\gamma(\gamma+2)V_*}{3\mathsf{s}} t \right]^{1/\gamma+2}. \qquad (38)$$

When t goes large, the $\ddot{\varphi}$ term will be less and less important because $\ddot{\varphi}/\dot{\varphi} \propto 1/t$ and so the slow-roll solution is ever improved. Also, again the energy density in the scalar field φ decays in time so that its fractional energy density always decreases and it will never dominate the total energy density. Equation (38) indicates that φ will continue to grow in the Λ-dominated era; however the rate of growth is lower than that in the case of $a \propto t^n$ [c.f. Eq. (35)], and can be very low when $\gamma \to \infty$. If needed, we could improve Eq. (38) by adding an asymptotic series, of which the leading terms are

$$\varphi^{\gamma+2} \sim \frac{\gamma(\gamma+2)V_*}{3\mathsf{s}} t + \frac{\gamma(\gamma+1)V_*}{9\mathsf{s}^2} \log t + \text{const} + \mathcal{O}\left(\frac{\log t}{t}\right).$$

VI. THE GENERAL CASE OF $\zeta \neq 0$ AND $V \neq 0$

Our ultimate aim is to consider the evolution of the scalar field φ when both the bare potential (c.f. Sec. IV) and the matter coupling term (c.f. Sec. III) are present, from which we can learn how the fine-structure constant α evolves in time. To that end we draw the contents of the above two sections together to get a picture of the whole

evolution of φ in the presence of both terms. We consider again the two cases: an exponential potential and an inverse power-law potential.

A. Exponential potential

Before going to the general radiation and matter-dominated solutions for the exponential potential and a matter coupling term, we first consider the special case that arises when $\lambda = 8$. The scalar-field equation now becomes Eq. (10)

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V(\varphi)}{\partial \varphi} = 2\frac{|\zeta|}{\omega}\rho_m e^{-2\varphi} \qquad (39)$$

and we still use the exponential potential given above.

We can see that Eqs. (17)–(20) remain as in the radiation-dominated solution. This is easy to check because for $a \propto t^{1/2}$, we have $\rho_m \propto a^{-3} \propto t^{-3/2}$, $e^{-2\varphi} = (e^{-\lambda\varphi})^{2/\lambda} \propto t^{-4/\lambda} = t^{-1/2}$, so α grows as $\exp(2\varphi) \propto t^{1/2}$, and both sides of Eq. (39) scale as t^{-2}. In this special case the presence of the coupling term does not influence the overall form of the solution, although Eqs. (22) and (23) might be changed (slightly) so that the (constant) fractional energy density Ω_φ is shifted in value. For the case $\lambda = 8$, originally we had $\Omega_\varphi = 1/16$ when $\varsigma = 0$, and in the radiation-dominated era $\rho_m \ll \tilde{\rho}_r$ so we expect the shift to be tiny.

This discussion can be generalized to include a subsequent era dominated by a fluid with general equation of states ($p \neq 0$). However, for a matter-dominated era, Eqs. (28)–(31) no longer remain an exact solution unless $\lambda \to \infty$.

Let us now turn to the more general $\zeta \neq 0$ cases. The effective total potential for the scalar field φ consists of two parts, the bare potential, $V(\varphi)$ and the electromagnetic matter couplings, which depends on $a(t)$, via $\rho_m \propto a^{-3}$, so we can combine them as

$$V_{\text{eff}}(\varphi) = V(\varphi) + \frac{|\zeta|}{\omega}\rho_m \exp(-2\varphi). \qquad (40)$$

The parameters used in this section are not specifically chosen to reproduce the observed time variation of the fine-structure constant (which we defer to the next section), rather here we are concerned with the general dynamics under the effective potential, which might also be useful for models of a scalar field with self-potential coupling to dark matter.

1. Radiation-dominated era

No matter which of the two parts to the scalar effective potential V_{eff} dominates, the field φ will grow at most logarithmically in the radiation era, and (except $\lambda = 8$) the only difference between the bare-potential-dominated and the coupling-dominated solutions is the coefficient in front of $\log t$. However, if that coefficient is very small then

φ will remain approximately constant during the radiation era, as in BSBM with $V = 0$ discussed above.

As it is the radiation era, if the scalar field makes a significant contribution to the energy budget of the Universe then $\rho_\varphi \sim V(\varphi) \gg \frac{|\zeta|}{\omega}\rho_m \exp(-2\varphi)$, (notice also that $\frac{|\zeta|}{\omega} \ll 1$ which reduces the possible influence of the coupling terms even further), and thus $V_{\text{eff}}(\varphi) \sim V(\varphi)$. On the other hand, if the scalar field constitutes only a very small part of the total energy density (for example, if $\lambda \gg 1$), then $V(\varphi)$ might be comparable to, or even much smaller than, $\frac{|\zeta|}{\omega}\rho_m \exp(-2\varphi)$.

Since the scale factor evolves as $a \propto t^{1/2}$ whichever part of the effective potential dominates, we shall take this as the leading-order solution to the Friedmann equation and look at the evolution of φ under this condition. We then could rewrite the scalar field equation of motion as

$$e^{-x}\frac{d}{dx}\left[e^{1/2x}\frac{d}{dx}\varphi(x)\right] = N\exp[-2\varphi(x)] + W\exp\left(\frac{3}{2}x\right)$$
$$\times \exp[-\lambda\varphi(x)] \qquad (41)$$

where we have defined $x \equiv \log t$, with $N \equiv 2\frac{|\zeta|}{\omega}\rho_m a^3$ and $W \equiv \lambda V_0$ constants. Clearly, the larger N is, the easier it is for the coupling term on the right-hand side to dominate, and the larger W is, the easier it is for the potential term to dominate.

As discussed above, we adopt a value of $\lambda > 2$; the larger λ is, the less important is the potential term for large φ. Because $\lambda > 2$, the potential term decays faster than the coupling term, so if the coupling term dominates at initial time, the potential term will never become important. On the other hand, if the potential term dominates at initial time, at some later time (if the radiation era is long enough) the effective potential will become dominated by the coupling term and the evolution of φ changes accordingly.

These behaviors are easy to verify numerically by solving Eq. (41), and an example is given in Fig. 1.

2. Matter-dominated era

Now we turn to the matter-dominated era in which $\rho_m \gg \tilde{\rho}_r$ and thus the latter can be neglected. If the bare potential is the major part of the effective potential then, according to previous analysis, the scale factor scales as $a \propto t^{2/3}$; if the coupling term dominates, Barrow *et al.* also showed [31–34,41] that $a \propto t^{2/3}$ in the leading-order solution. So here we also assume that this is a good approximation in the matter-dominated era and look at the evolution of φ on this background.

In this case the scalar field equation of motion becomes

$$e^{-x}\frac{d}{dx}\left[e^x\frac{d}{dx}\varphi(x)\right] = N\exp[-2\varphi(x)]$$
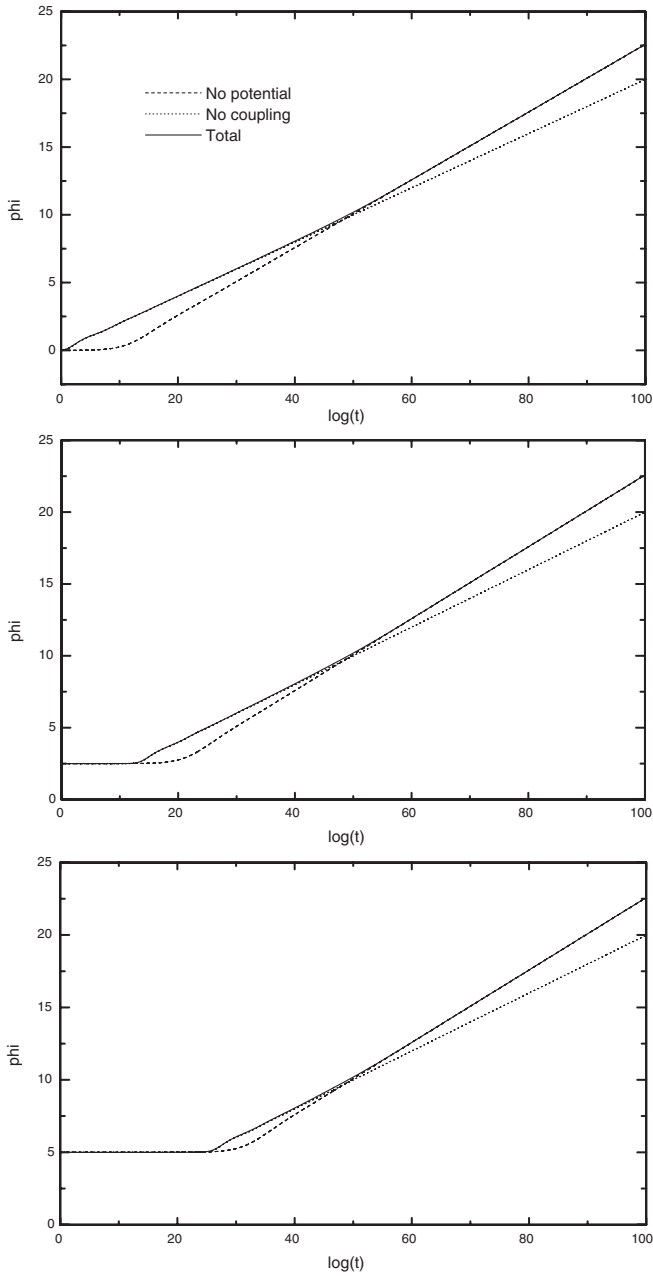$$+ W\exp(2x)\exp[-\lambda\varphi(x)]. \qquad (42)$$

FIG. 1. The evolution of φ as a function of $\log t$ in a radiation-dominated universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters are $\lambda = 10$, $N = 0.001$, and $W = 0.1$; the initial conditions in the upper, middle, and lower panels are $\dot{\varphi}_i = 0$ and $\varphi_i = 0$, 2.5, and 5, respectively.

term, then because the former scales as t^{-2} while the latter scales as $\rho_m \exp(-2\varphi) \propto t^{-2-4/\lambda}$ and falls faster, the potential term will become increasingly dominant and the coupling term will never become important. On the other hand, if initially the coupling term dominates, then the potential term and the coupling term scale as $(\log t)^{-\lambda/2}$
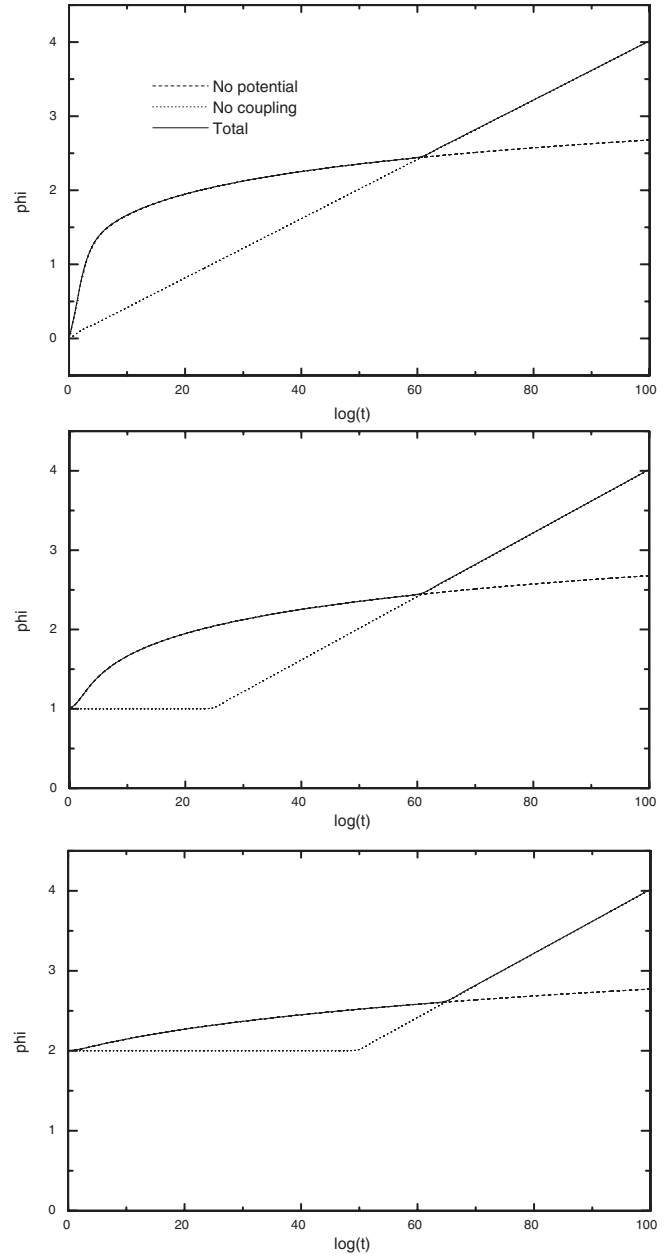

FIG. 2. The evolution of φ as a function of $\log t$ in a matter-dominated background universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters chosen are $\lambda = 50$, $N = 1$, and $W = 0.1$; the initial conditions in the upper, middle, and lower panels are $\dot{\varphi}_i = 0$ and $\varphi_i = 0$, 1, and 2, respectively.

A qualitative analysis can be made as in the case of radiation domination, by considering the evolution without the potential or coupling term present, respectively. In the case where only the potential term is included, we have $\varphi \propto \log t$; if only the coupling term is presented then φ evolves as $1/2 \log(2N \log t)$ approximately. Now suppose that initially the potential term dominates over the coupling

and $t^{-2}[\log(t)]^{-1}$, respectively, and the former always falls off slower than the latter (however, depending on the value of λ, the dominance of the potential term could occur very late, much later than the transition to an acceleration era, so probably we will not see this transition during the matter epoch). Thus, in this case, the potential term will finally overwhelm the coupling term, and the full solution will then track the no-coupling one. These features can also be checked numerically. Note that increasing N or λ will help the coupling term to dominate. The tracking is excellent in both regions. When the coupling term dominates the fine-structure constant evolves as $\alpha \propto 2N \log t$, but when the potential term dominates, $\alpha \propto (t/t_0)^{4/\lambda}$.

A numerical example of the behaviors discussed above is shown in Fig. 2.

3. Dark-energy-dominated era

A later times the Universe will become dominated by any cosmological constant Λ and then the scalar field equation of motion becomes

$$\ddot{\varphi} + 3 \varsigma \dot{\varphi} = W \exp(-\lambda \varphi) + N \exp(-3 \varsigma t) \exp(-2 \varphi).$$

If the evolution is initially dominated by the bare-potential term (W), then we know from the above analysis that $\varphi \sim \frac{1}{\lambda} \log t$. So the bare-potential term in the above equation scales as $1/t$ while the coupling term scales exponentially with respect to t, as $t^{-(2/\lambda)} \exp(-3 \varsigma t)$. For large t, the latter decays faster, and the bare-potential term will eventually dominate over the coupling term and the fine-structure constant will evolve all the time in the future. A numerical example is shown in Fig. 3. Note that for this analysis we have defined a different set of parameters $x \equiv \log a$, with $W \equiv \lambda V_0/\varsigma^2$ and $N \equiv 2|\zeta|\rho_{m0}/\varsigma^2$ constants, so that the above equation of motion is rewritten as

$$\frac{d^2}{dx^2} \varphi(x) + 3 \frac{d}{dx} \varphi(x) = N \exp(-3x) \exp(-2\varphi)$$

$$+ W \exp(-\lambda \varphi).$$

If the dark energy drives power-law inflation $a \propto t^n$ ($n > 1$) of the Universe today, then the analysis of the whole evolution is qualitatively the same as for a radiation-dominated universe and will not be repeated here.

B. Inverse power-law potential

Next, we consider the evolution of φ under the controls of the coupling term and a bare inverse power-law potential.

1. Radiation-dominated era

In the radiation-dominated era the cosmic scale factor scales as $a \propto t^{1/2}$, while the energy densities of the scalar field and dust can be neglected. So, just as in the case of exponential potential, we can write the scalar field equation
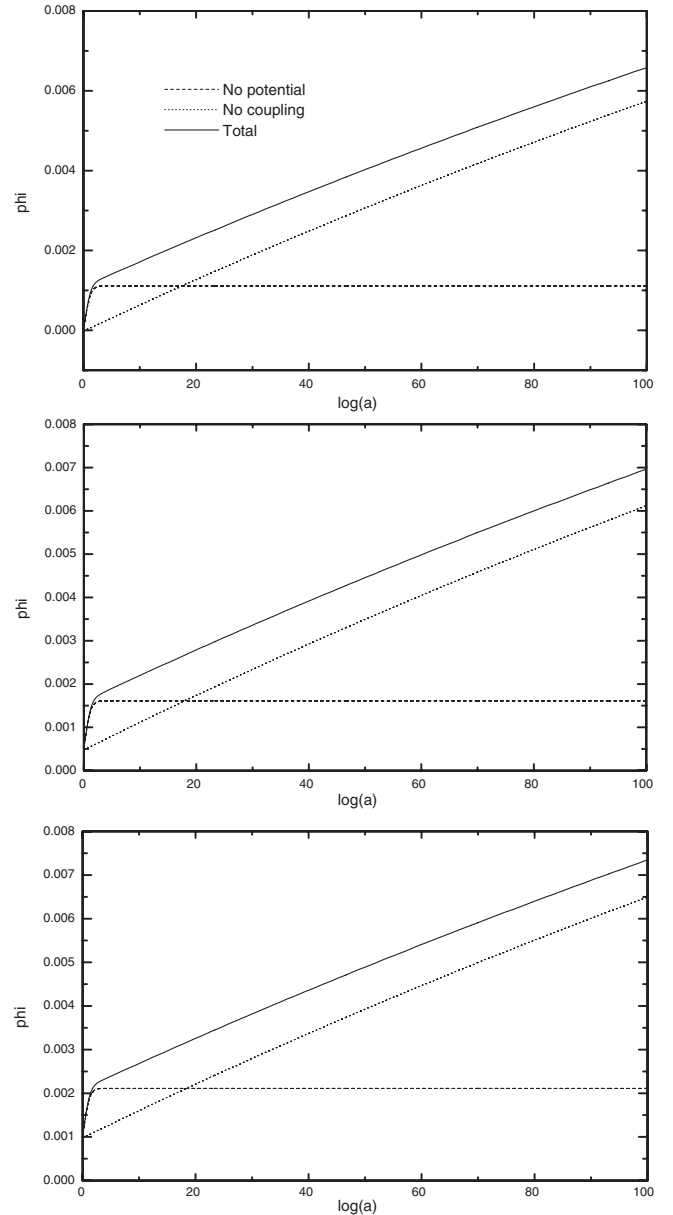

FIG. 3. The evolution of φ as a function of $\log a$ in a Λ-dominated universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters chosen are $\lambda = 50$, $N = 0.01$, and $W = 0.0002$; the initial conditions in the upper, middle, and lower panels are $\dot{\varphi}_i = 0$ and $\varphi_i = 0$, 0.0005, and 0.001, respectively.

of motion as

$$e^{-x} \frac{d}{dx} \left[e^{1/2x} \frac{d}{dx} \varphi(x) \right] = N \exp[-2\varphi(x)]$$

$$+ W \exp\left(\frac{3}{2} x\right) \varphi^{-(\gamma+1)}, \quad (43)$$

where again $x = \log t$, but now $W \equiv \gamma V_*$.

According the above analysis, if the bare-potential term dominates then φ evolves as $\varphi \sim t^{2/\gamma+2} \sim \exp(\frac{2}{\gamma+2}x)$, (so φ is exponential in x), while if the coupling term dominates then $\varphi \sim \frac{1}{4}\log t \sim \frac{1}{4}x$ (so φ is linear in x). These features can be seen clearly in the numerical example given in Fig. 4. Note that in the scalar field equation of motion the bare-potential term decays as $\sim \varphi^{-(\gamma+1)}$, while the cou-

pling term scales like the terms $\sim \rho_m \exp(-2\varphi) \sim t^{-(3/2)} \exp(-2\varphi) \sim \varphi^{-(3/4)(\gamma+2)} \exp(-2\varphi)$, (when the effective potential V_{eff} is dominated by the bare potential), or as $\sim \rho_m \exp(-2\varphi) \sim t^{-(3/2)} \exp(-2\varphi) \sim \exp(-8\varphi)$, (when V_{eff} is dominated by the coupling term), so obviously in both cases when eventually φ is large enough the
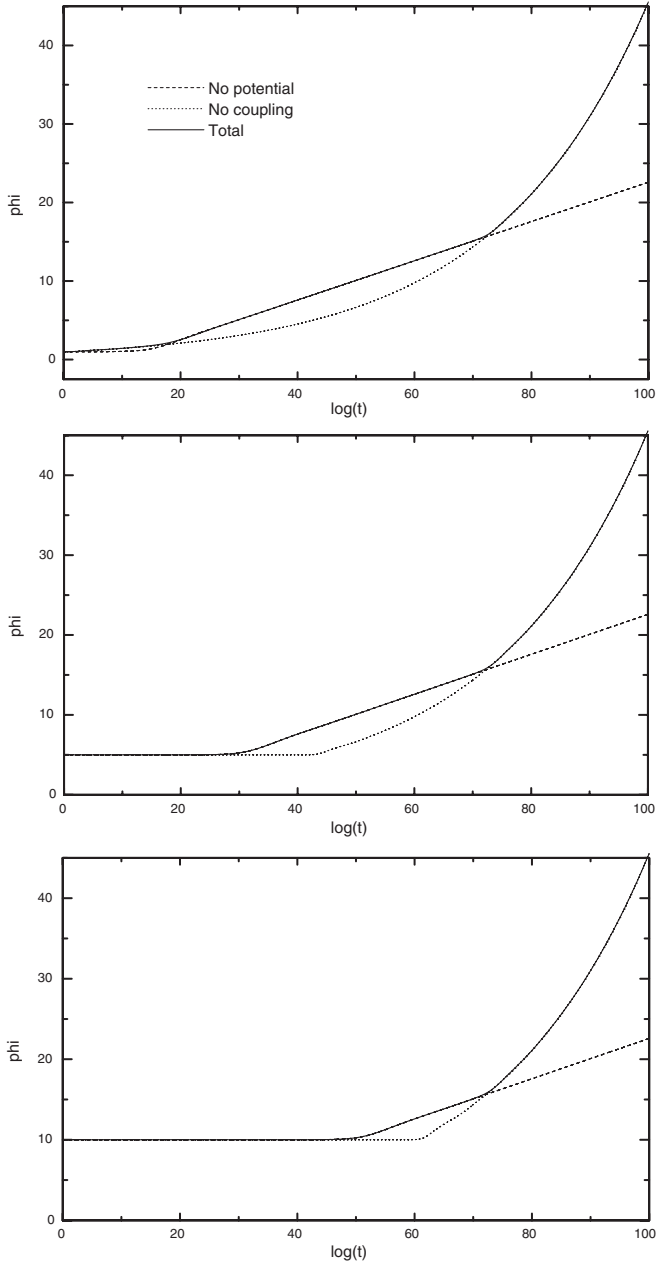

FIG. 4. The evolution of φ as a function of $\log t$ in a radiation-dominated universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters used are $\gamma = 50$, $N = 0.001$, and $W = 0.005$; the initial conditions in the upper, middle, and lower panels are $\dot{\varphi}_i = 0$ and $\varphi_i = 1$, 5, and 10, respectively.
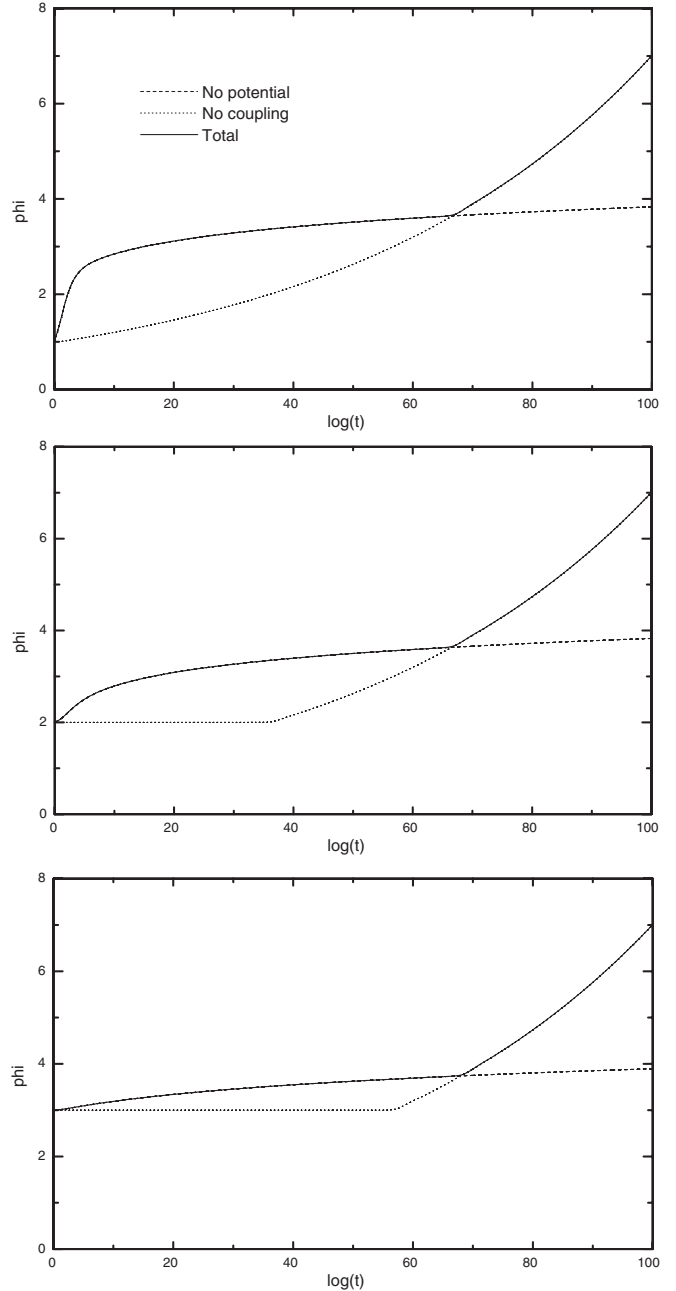
FIG. 5. The evolution of φ as a function of $\log t$ in a matter-dominated universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters chosen are $\lambda = 100$, $N = 10$, and $W = 0.005$; the initial conditions in the upper, middle, and lower panels are $\dot{\varphi}_i = 0$ and $\varphi_i = 1$, 2, and 3, respectively.
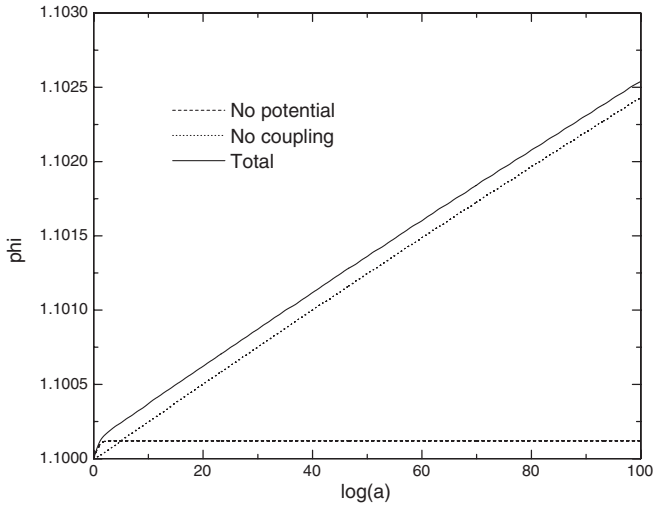
FIG. 6. The evolution of φ as a function of $\log a$ in a Λ-dominated universe. The solid, dashed, and dotted curves represent the total evolution, the evolution governed solely by the coupling term, and that governed only by the bare-potential term, respectively. The parameters are $\gamma = 50$, $N = W = 0.01$, and the initial conditions are $\dot{\varphi}_i = 0$, and $\varphi_i = 1.1$.

bare-potential term will dominate over the coupling term driving the evolution of φ. This can also be seen in the figure.

2. Matter-dominated era

In the matter-dominated era the scale factor evolves as $a \propto t^{2/3}$, while the energy densities of the scalar field and radiation can be neglected. In this case the scalar field equation of motion becomes

$$e^{-x} \frac{d}{dx}\left[e^x \frac{d}{dx} \varphi(x) \right] = N \exp[-2\varphi(x)]$$
$$+ W \exp(2x)\varphi^{-(\gamma+1)}, \quad (44)$$

where x and W are as defined above.

From the above analysis we know that if the bare-potential term dominates then φ evolves as $\varphi \sim t^{2/\gamma+2} \sim \exp(\frac{2}{\gamma+2}x)$ (i.e., φ is exponential in x), while if the coupling term dominates then to the leading order $\varphi \sim \frac{1}{2} \log(\log t) \sim \frac{1}{2} \log x$ (i.e., φ is logarithmic in x). We can also see these behaviors clearly in the numerical results plotted in Fig. 5.

In the scalar field equation of motion the bare-potential term decays as $\sim \varphi^{-(\gamma+1)}$, while the coupling term scales either as $\sim \rho_m \exp(-2\varphi) \sim t^{-2}\exp(-2\varphi) \sim \varphi^{-(\gamma+2)}\exp(-2\varphi)$, (when $V_{\rm eff}$ is dominated by the bare potential), or as $\sim \rho_m \exp(-2\varphi) \sim t^{-2}\exp(-2\varphi) \sim \exp[-2\exp(2\varphi)]\exp(-2\varphi)$ (when $V_{\rm eff}$ is dominated by the coupling). Therefore, when φ eventually grows large enough, the bare-potential term will dominate over the coupling term and drive the evolution of φ. This is verified in the numerical results, where we can see specific tracking solutions in different epochs.

3. Dark-energy-dominated era

In a universe dominated by a cosmological constant, the field equation for φ in the case of a power-law potential can be written as

$$\frac{d^2}{dx^2}\varphi(x) + 3\frac{d}{dx}\varphi(x) = \frac{W}{\varphi^{\gamma+1}} + N\exp(-3x)\exp(-2\varphi),$$

where $x \equiv \log a$, $W \equiv \gamma V_*/\varsigma^2$, and $N \equiv 2|\zeta|\rho_{m0}/\varsigma^2$.

According to the above analysis, when the effective potential $V_{\rm eff}$ is dominated by the coupling term, φ will approach a constant in the Λ-dominated era, while if $V_{\rm eff}$ is dominated by the bare potential then φ evolves as $\varphi \sim t^{1/\gamma+2} \propto (\log a)^{1/\gamma+2}$. These qualitative behaviors can be observed in Fig. 6. As φ goes large, the bare-potential term in $V_{\rm eff}$ decreases as $\sim \varphi^{-(\gamma+1)}$ while the coupling term decays as $\sim \exp(-3\varsigma t)\exp(-2\varphi)$, so eventually the former will always dominate over the latter, driving the continuous growth of φ in contrast to the asymptotically constant behavior seen in BSBM where $W = 0$.

C. Summary of the behaviors of $\varphi(t)$ and $\alpha(t)$

We summarize the possible behaviors of $\varphi(t)$ found in the different varying-α models that we have discussed, in the three different cosmic eras and different situations (bare potential $V_{\rm eff}$ dominated or coupling ζ dominated) in Table I. The time evolution of α is obtained from that of φ by using $\alpha \propto \exp(2\varphi)$. Recall from the figures above that the transitions from bare-potential domination to coupling domination (or vice versa) are smooth, so there is a simple pattern for the overall φ evolution. However, note that depending on the initial conditions for φ, the above "tracking" solutions may only be reached after a long time

TABLE I. Tracking behaviors of α in different varying-α models in different limits. V denotes the bare-potential term and C is the coupling term; $\rightarrow$ means an asymptotic approach.

Model	Radiation Era	Matter Era	Acceleration Era
BSBM	$\alpha \sim t^{1/2}$	$\alpha \sim \log t$	$\alpha \rightarrow$ const
BSBM $+ V_0 \exp(-\lambda\varphi)$	$\alpha \sim \begin{cases} t^{4/\lambda}, & V \text{ dom} \\ t^{1/2}, & C \text{ dom} \end{cases}$	$\alpha \sim \begin{cases} t^{4/\lambda}, & V \text{ dom} \\ \log t, & C \text{ dom} \end{cases}$	$\alpha \begin{cases} \sim t^{4/\lambda}, & a \propto t^n \\ \rightarrow t^{2/\lambda}, & H = \text{const} \end{cases}$
BSBM $+ V_* \varphi^{-\gamma}$	$\alpha \sim \begin{cases} \exp(2t^{2/\gamma+2}), & V \text{ dom} \\ t^{1/2}, & C \text{ dom} \end{cases}$	$\alpha \sim \begin{cases} \exp(2t^{2/\gamma+2}), & V \text{ dom} \\ \log t, & C \text{ dom} \end{cases}$	$\alpha \begin{cases} \sim \exp(2t^{2/\gamma+2}), & a \propto t^n \\ \rightarrow \exp(2t^{1/\gamma+2}), & H = \text{const} \end{cases}$

(as can be seen in the figures). In this analysis we have focussed upon extracting the evolution of $\varphi(t)$, and hence of $\alpha(t)$, but a more detailed study with a different emphasis could also seek the best-fit observational parameter set when varying alpha are included, as was done in Ref. [47].

VII. NUMERICAL EXAMPLES OF THE φ AND α EVOLUTION

We shall now consider the numerical evolution of φ and α through the entire cosmological history, and try to connect this evolution to the observations constraining possible time variation in α. As we have seen above, the evolution in φ is controlled by the competition between the coupling term and bare-potential term in the effective potential. The parameter $|\zeta|/\omega$ determines the strength of coupling and so increasing it will increase the rate of variation of φ. In addition, in the case of the exponential potential, $V = V_0 e \exp(-\lambda \varphi)$, the parameter λ controls the scaling solution of φ, and the larger it is, the smaller the fraction of the total energy density Ω_φ tends to be; note also that λ governs the slope of the evolution of φ : if there is no coupling term then the solution of $\alpha \propto \exp(2\varphi)$ is given by $\alpha \propto a^{-(8/\lambda)}$ in the radiation era and $\alpha \propto a^{-(6/\lambda)}$ in the radiation era. This power-law evolution means that in order that the fractional variation of α between now and $z \sim 6$ should not exceed the observational bounds, i.e., $\Delta\alpha/\alpha < \mathcal{O}(10^{-5})$, λ must be very large. Similarly, in the case of an inverse power-law potential, γ controls the scaling solution of φ, and larger values of γ correspond to smaller variations of φ in time, so that an observational constraint that the allowed variation of $\alpha \propto \exp(2\varphi)$ be small requires γ to be very large.

Finally, the initial value φ_i, is also an important quantity; although for the exponential potential one can always choose $\varphi_i = 0$ by adjusting V_0 correspondingly. For the coupling term we do not have this freedom—a larger φ_i will weaken the coupling through $\exp(-2\varphi_i)$ and thus reduce the change in φ.

In Figs. 7 and 8 we have plotted the evolution of $\varphi(t)$ for some choices of the model parameters for the two above-mentioned potentials, respectively. For comparison, we also plot the result for the BSBM model (where $V = 0$) with the same choice of the parameter $|\zeta|/\omega$. In both cases φ is initially dominated by the bare-potential term but later becomes dominated by the coupling term because the largeness of λ (or γ) means that the slope of the bare-potential-dominated evolution is smaller that the coupling-dominated evolution at later matter-dominated times. When the Universe becomes dominated by the cosmological constant, the coupling-dominated solution for φ approaches a constant and the bare-potential-dominated solution grows very slowly, so the total solution grows very slowly too (almost constant in the figure). Note that the addition of a bare-potential term makes φ begin evolving earlier. This produces an earlier onset of variation for α
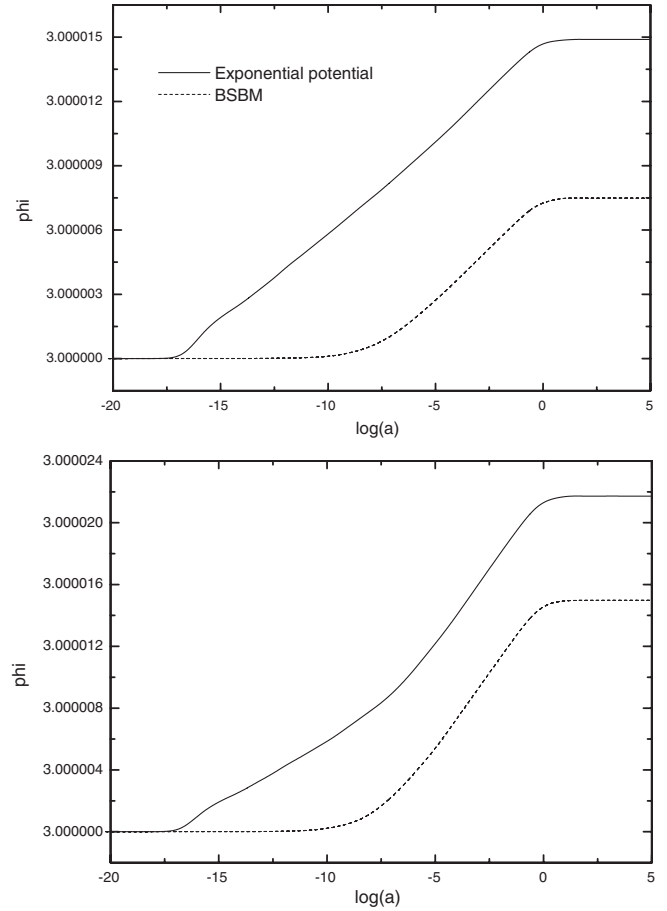

FIG. 7. Comparison of the entire cosmological evolution of φ as a function of $\log a$ in the BSBM model (dashed curve) and the model with an exponential potential $V = V_0 \exp(-\lambda \varphi)$ plus coupling term (solid curve). The parameters for the upper panel are, respectively, $\lambda = 5 \times 10^6$, $|\zeta|/\omega = 10^{-4}$ and $\lambda = 5 \times 10^6$, $|\zeta|/\omega = 2 \times 10^{-4}$. In both cases the initial conditions are $\varphi_i = 3$, $\dot{\varphi}_i = 0$.

than in the pure BSBM model (the late-time evolution, which is relevant for the quasistellar object observations, is however almost the same as in BSBM because at this late stage the total solution is dominated by the coupling term).

Figures 9 and 10 show the evolution of α/α_0 (where α_0 is the current value of the fine-structure constant) in the two models when compared with the prediction of the BSBM model (dashed curves). The qualitative feature in a model with self-potential for φ is the same as in BSBM: at early times α remains a constant; during the matter-dominated era there is a slow growth; and then, when the cosmological constant begins to dominate, the growth stops. The differences are: first, the commencement of growth for α starts earlier than the BSBM case because in this model the bare-potential term could drive the evolution of φ even in the radiation-dominated era; second, the late-time evolution is dominated by the coupling term and so the late-time evolution of φ is similar to that in BSBM, but the earlier commencement of the growth means that the total variation
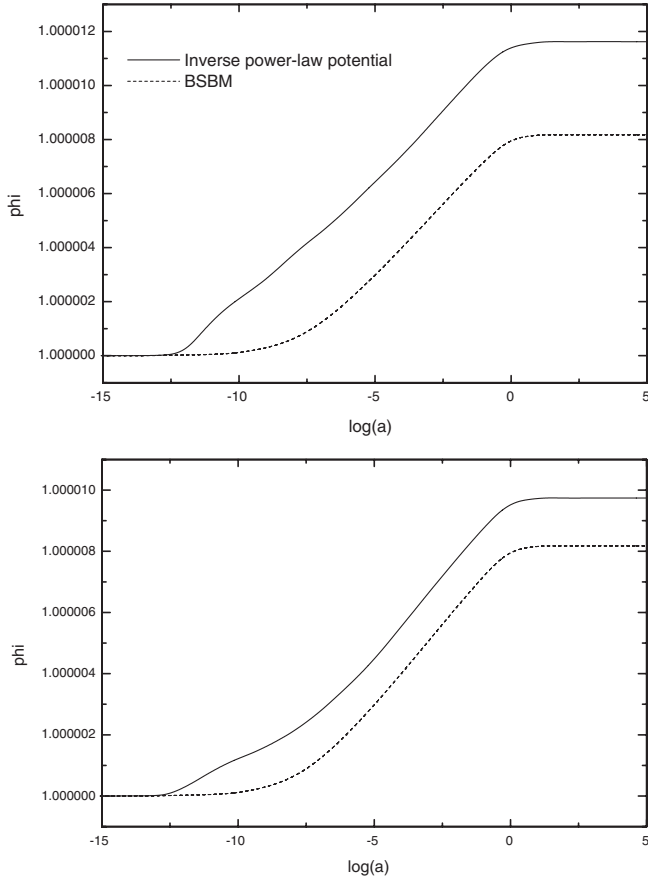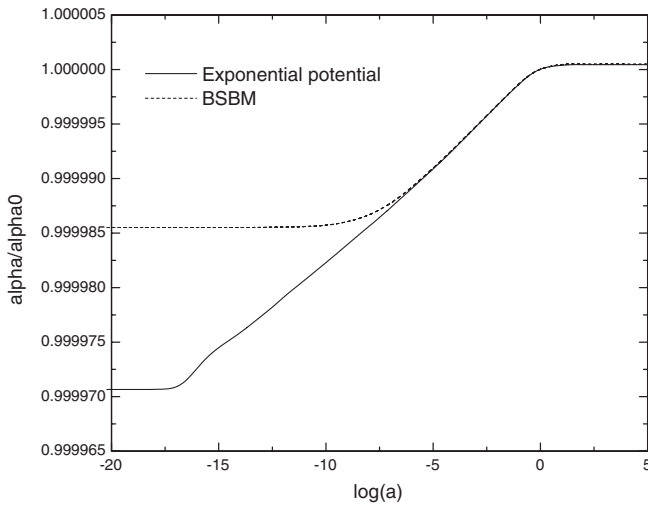
FIG. 8. Comparison of the entire cosmological evolution of φ as a function of $\log a$ in the BSBM model (dashed curve) and the model with an inverse power-law potential $V = V_* \varphi^{-\gamma}$ plus coupling term (solid curve). The parameters for the upper panel are, respectively, $\gamma = 5 \times 10^6$, $|\zeta|/\omega = 2 \times 10^{-6}$ and $\gamma = 1 \times 10^7$, $|\zeta|/\omega = 2 \times 10^{-6}$. In both cases the initial conditions are $\varphi_i = 1$, $\dot{\varphi}_i = 0$.

FIG. 9. The evolution of α/α_0 versus $\log a$ in the BSBM model (dashed curve) and in the model with an exponential potential plus coupling term (solid curve). The parameters chosen are $\lambda = 5 \times 10^6$, $|\zeta|/\omega = 10^{-4}$.
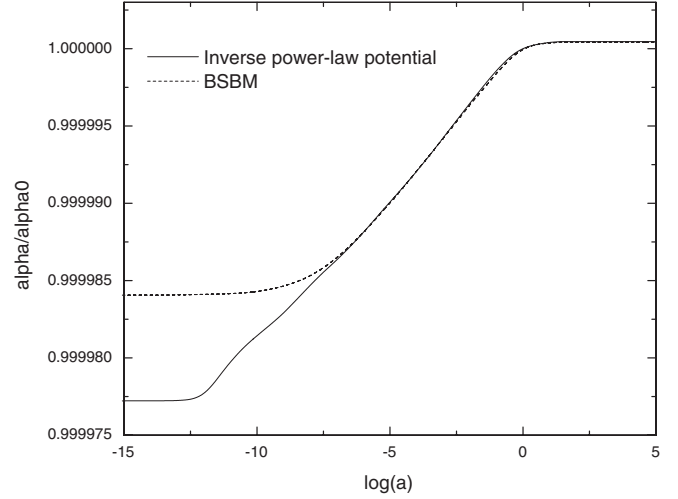

FIG. 10. The evolution of α/α_0 versus $\log a$ in the BSBM model (dashed curve) and in the model with an inverse power-law potential plus coupling term (solid curve). The parameters chosen are $\gamma = 5 \times 10^6$, $|\zeta|/\omega = 2 \times 10^{-6}$.

of α is greater than that in BSBM. The fractional change of α between $z \sim 4$ and now is about 0.5×10^{-5} which is consistent with the quasistellar object observations. The current rate of $\dot{\varphi}$ is given by $\dot{\varphi}_0 = (\frac{d\varphi}{dx})_0 H_0$ where $H_0 \approx 70$ kms^{-1} Mpc^{-1} is the present Hubble expansion rate. For the parameters in Fig. 9, we have $(\frac{d\varphi}{dx})_0 \sim 0.8 \times 10^{-6}$ and so we have $\dot{\varphi}_0 \sim 0.6 \times 10^{-16}$/yr which leads to $(\dot{\alpha}/\alpha)_0 \sim 2\dot{\varphi}_0 \sim 1.2 \times 10^{-16}$/yr. This rate is well within all old limits [6–9] but is about an order of magnitude above the proposed new upper bound [19] on the current rate of α variation from atomic clocks, which is $(\dot{\alpha}/\alpha)_0 = (-1.6 \pm 2.3) \times 10^{-17}$/yr, although the uncertainties may be modulated slightly by accounting for seasonal variations in the local gravitational potential [20].

VIII. SUMMARY AND CONCLUSIONS

In this paper we have considered the dynamics of the varying-α theories which arise when the original BSBM theory is generalized by introducing an exponential or inverse power-law self-potential for the scalar field driving the variation of α. These two representative potentials capture the essential ingredients of general potentials without minima. There are two situations to distinguish and analyze separately: according as to whether or not the scalar-field potential comes to dominate the late-time dynamics of the Universe. In combination with the coupling with matter, the additional bare potential forms an effective total potential V_{eff} for the scalar field φ which governs the allowed time variation of α. We have presented the solutions to the scalar-field equation of motion in cases where V_{eff} is dominated solely by the coupling term or the bare potential, respectively, in different cosmic eras. In most cases the bare-potential-dominated solution differs from

the coupling-dominated one; the contributions of these two terms to V_{eff} vary with time, and it is possible for there to be a transition from one solution type to another. The numerical results show that the transition between solutions types can be very smooth, and for most of the time the solution for the scalar field tracks either the bare-potential-dominated or the coupling-dominated solution. These features ensure that the evolution of φ under V_{eff} has a very simple pattern. The main results are briefly summarized in Table I, and these results are quite general, not depending on whether the parameters defining the potential (λ and γ) are extremely large or not.

The consequences for the time evolution of the fine-structure constant of adding potentials $V(\varphi)$ to the BSBM theory are summarized as follows. In light of the observational constraints on how much variation in α there can be over redshifts $z < 6$, we find that the restrictions on the interaction potential parameters (λ in the exponential potential and γ in the inverse power-law potential) must be very strong, in order to prevent the bare potentials from becoming unacceptably dominant and driving unacceptable fast time variation of α. For example, in the case of an exponential potential, $V = V_0 \exp(-\lambda\varphi)$, we must have $\lambda \gtrsim 10^6 \sim 10^7$ with $|\zeta|/\omega \sim \mathcal{O}(10^{-6})$, and for the inverse power-law potential, $V = V_*\varphi^{-\gamma}$, we need $\gamma \gtrsim 10^6 \sim 10^7$ with $|\zeta|/\omega \sim \mathcal{O}(10^{-6})$ (the exact constraint, of course, depends also on the initial conditions). This means that the λ and γ are constrained to be so large that if they appear in quintessence models then the scalar field is practically indistinguishable from a cosmological constant, which has no dynamics at all. The total variation of α can be enhanced compared to the case of no bare potential (BSBM), and the variation commences much earlier. Finally, because with an exponential or inverse power-law potential the scalar field will not approach a constant even in the dark-energy-dominated era, the fine-structure constant α will continue to increase for all future time, until eventually there will no stable atoms in the Universe at all [32].

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