Feynman's *different* approach to electrodynamics

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Abstract

We discuss a previously unpublished description of electromagnetism developed by Richard P. Feynman in the 1960s. Though similar to the existing approaches deriving electromagnetism from special relativity, the present one extends a long way towards the derivation of Maxwell's equations with minimal physical assumptions (in particular, without postulating Coulomb's law). Homogeneous Maxwell's equations are, indeed, derived by following a route different to the standard one, i.e. first introducing electromagnetic potentials in order to write down a relativistic invariant action, which is just the inverse approach to the usual one. Also, Feynman's derivation of the Lorentz force exclusively follows from its linearity in the charge velocity and from relativistic invariance. The obvious historical significance of such approach is then complemented by its possible relevance for didactics, providing a novel way to develop the foundations of electromagnetism, which can fruitfully supplement usual treatments.

One man's assumption is another man's conclusion [1]

1 Introduction

It is a well-known story that Richard P. Feynman was not satisfied with his presentation of electricity and magnetism in his second year general physics lectures, which he gave at Caltech in 1961-63. In his preface to the published version [2] he wrote:

> In the second year I was not so satisfied. In the first part of the course, dealing with electricity and magnetism, *I couldn't think of any really unique or different way of doing it* – or any way that would be particularly more exciting than the usual way of presenting it. So I don't think I did very much in the lectures on electricity and magnetism.

As a matter of fact, indeed, the way of presenting electromagnetism in the second volume of Feynman's Lectures [2] is quite similar to many existing books on the subject at various level – one for all [3] – where a historical/experimental path starting with electrostatics is followed, then moving on to magnetostatics, Faraday's law of induction, Maxwell's equations, electromagnetic waves and, finally, showing the consistency of Maxwell's equations with the special theory of relativity (which is however introduced in the first volume of Ref. [2]).

Later on, in 1966, in an interview for the American Institute of Physics with Charles Weiner [4][1], Feynman said:

> *Now I think I know how to do it.* [...] *I've now cooked up a much better way of presenting the electrodynamics, a much more original and much more powerful way than is in the book.*

Unfortunately, however, no such presentation appears to have been published or used by Feynman.

Motivated by the natural curiosity aroused by such claims, M.A. Gottlieb sought for an answer in the Caltech archives, where he discovered five pages of handwritten notes, dating from 13 December 1963, in which Feynman sketched his ideas. The notes (both the originals and a transcript by Gottlieb himself) were made available online [6]. To the best of our knowledge, no attempt to explain the physics contained in those notes was made by Gottlieb or anyone else up to now, and thus, the aim of the present paper is to study and clarify these notes, in order to make their content usable. We believe that, besides the obvious historical interest, this work can be useful for teaching purposes, since it suggests an alternative view on at least part of the fundamentals of electromagnetism.

Feynman's notes contain an outline of a possible course on electromagnetism, along with some reflections on possible disadvantages and advantages (interestingly, the disadvantages come first), and a sketch of a derivation of the Lorentz force law from the requirements of special relativity and of charge invariance, which is the first point in the outline.

The idea of deriving electromagnetism from relativity (rather than following the inverse, historical route) is not new, dating back at least to 1912 [7]. The idea is to start from Coulomb's law and then derive Lorentz force from Lorentz transformations. While mentioned in most textbooks about electromagnetism, this viewpoint has been chosen as the *true* way to proceed in many other textbooks and papers, see e.g. [8, 9, 10, 11, 12, 13, 14, 15] and references therein.[2] Some of these references content themselves with the derivation of magnetostatics from electrostatic and relativity, while others try to get the full electrodynamics by extending to accelerated charges. Very interestingly, this same viewpoint has been adopted by Feynman in teaching Electromagnetism at the Hughes Aircraft company in 1967-68 [1], and some comment about it is also included in the chapter about magnetostatics in the second volume of the lectures.

While sharing with the above viewpoint the idea of deriving electromagnetism from relativity (rather than the opposite), Feynman's approach sketched in his 1963 notes is radically different. Feynman, indeed, derives the form of the interaction (i.e. Lorentz force) just from the requirements of

[1]The relevant part of that interview is also reported in [5].

[2]Such an approach, in order to be logically consistent, would of course require the introduction of special relativity independently of electromagnetism, which is indeed possible. For example, it was used by Feynman himself in the first volume of his lectures, or in many of the references cited in the main text. Moreover, it was explored in many other references about special relativity, such as Ref. [16], where special relativity is developed from the idea that any interaction should have a limiting speed, only later to be identified with the speed of light, and Ref. [17], where the relativistic law of addition of parallel velocities is derived only from the first of Einstein's postulates, while not invoking the second one.

charge invariance (coming from experiments), relativity and linearity. Such a derivation, to the best of our knowledge, is absent in the literature, and was not even used by Feynman in later courses, including the Hughes lectures. This was probably due to the fact that he did not continue along this line of reasoning in the derivation of Maxwell's equations and, more in general, of the full electrodynamics. Nevertheless, some material in the Hughes Lectures, as we shall see, might be in line with Feynman's ideas in these notes, since, after a long introduction about the least action principle (notoriously one of Feynman's favorites), he described a relativistically invariant generalization of it where force is implemented by the addition of a 4-vector potential. Upon comparison of the resulting equations of motion with the Lorentz force – which may well have been derived using the approach sketched in the notes, Feynman was able to find the link between the electric and magnetic fields and the components of the 4-potential. These expressions, upon using well known vector analysis identities, led to the homogeneous Maxwell equations. Moreover, by allowing different potentials while requiring the invariance of the classical equations of motions resulting from minimizing the action, Feynman was able to introduce gauge transformations from perspective different than the usual one. By combining Feynman's approach to the Lorentz force with his derivation of the homogeneous Maxwell's equations, it is then possible to go quite a long way towards the full derivation of electromagnetism, just with a minimal physical input.[3]

2 Deriving the Lorentz force

In the first page of his notes, Feynman outlined a possible full course on electrodynamics, articulated in seven points as follows:

1. Get the Lorentz force equation from charge conservation and relativity, get the potentials or at least the homogeneous Maxwell equations.

2. Discuss the field idea, discuss the qualitative properties and shapes of the fields in some situations, differential operators, motion of electrons on given fields, induction, etc.

3. Get the inhomogeneous Maxwell's equations, either by arriving at the wave equation $\Box \mathbf{A} = \mathbf{j}$ for the potentials from relativity or from some other principle, or even through the experimental, usual route (Coulomb, Ampère, ...).

4. Discuss the fields produced in several semi-static circumstances, e.g. condensers, inductances, etc.

[3] Interesting enough, this was not the only time Feynman entertained himself with original presentations of electromagnetism. As reported by Freeman Dyson (quoted in [18]), in 1948 he was able to derive the homogeneous Maxwell's equations solely from Newton's law and from the canonical Poisson brackets (or even canonical commutation relations) for a non-relativistic particle; such a derivation was much later published by Dyson himself [19], and was soon recognized as a particular case of inverse variational problem (see e.g. [20] and references therein). Notice that Feynman's proof of 1948 is given in a Galilei invariant context, while the proof we are studying here is given in a Lorentz invariant context. However, there is no contradiction since both proofs concern only the Lorentz force and the homogeneous Maxwell equations, which are invariant under both groups (the deep origin of this is that the homogeneous Maxwell equations are topological in nature, since they do not depend on the metric, see e.g. [21]).

5. Discuss the field in wave situations, e.g. radiation, waveguides, etc.

6. Energy and related problems, self-mass, Lienard-Wiechert potential, etc.

7. Fields in matter: **D**, **H**, etc.

Points 1 and 3 are, of course, the crucial ones, as pointed out by Feynman himself, while the other ones only concern applications of electrodynamics, rather than the formulation of electromagnetic laws, so that it is very much plausible that he aimed to address them in a way similar to the one adopted in the Caltech or Hughes Lectures. Remarkably, however, in his notes Feynman only addressed the first part of point 1, relegating the second part just to a question (posed but not answered, at least not in the notes): "Can you find other ways? – Can you get potentials in?". Point 3 is not addressed as well.

Feynman's approach to electromagnetism can be traced back to the fundamental assumption that matter has an atomic structure (a well-known citation shows that he considered this to be the single most important scientific fact (as stated at the very beginning of [2]).

2.1 Linear dependence on the velocity

The starting point, as delineated in the fifth page of his notes, is that motion does not alter charge, i.e. electric charge is Lorentz invariant. The usual argument for this, followed by Feynman too, went as follows. Atoms are electrically neutral, but electrons in them move, even at very high speeds, this meaning that the electron charge is not affected by such motion. This is in contrast with what happens for mass-energy, which does depend on the velocity. Here Feynman referred properly to relativistic energy, which may be described in terms of a velocity-dependent "relativistic mass", considered as a gravitational mass (i.e. weight) rather than inertial mass, though the two coincide due to the equivalence principle. For atoms, this is exemplified by the fact that an excited atom weights more than an atom in its ground state, and Feynman also referred to a possible experimental proof of this fact, while not explicitly mentioning it[4].

Having established Lorentz invariance for the electric charge, the next step was the observation that a conducting wire where an electric current is present, is electrically neutral. If two such wires are placed next to each other, however, a net force between them is observed (and here probably Feynman planned a demonstration of this effect): this suggests that the force between electric charges depends not only on their position, but also on their velocity. Therefore, we conclude that a point charge q, moving anyhow in the presence of other charges, experiences a force $\mathbf{F}$ which is a function of its position and velocity:

$$\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{v})\,. \tag{1}$$

[4] It is worth noting that this same line of reasoning was followed by Feynman in [22], in order to argue that gravitation must be mediated by a spin-2 field rather than a spin-0 field, whose charge would decrease with velocity, instead of increasing.

Charge invariance can be used again to observe that, for an oscillating charge, i.e. a charge moving in such a way that its average velocity is zero, the average force $\overline{\mathbf{F}(\mathbf{x},\mathbf{v})}$ acting on it must be equal to the force acting on a charge at rest, i.e.

$$\overline{\mathbf{F}(\mathbf{x},\mathbf{v})} = \mathbf{F}(\mathbf{x},\mathbf{0}) \qquad \text{if} \qquad \overline{\mathbf{v}} = 0\,. \tag{2}$$

This fact necessarily implies that force depends *linearly* on velocity,[5] that is $F_i(\mathbf{x},\mathbf{v}) = a_i + b_{ij}v_j$, where a_i, b_{ij} are coefficients which can be cast in the more expressive following form (making explicit the electric charge q):

$$F_i(\mathbf{x},\mathbf{v}) = q(E_i + v_j\, B_{ij}), \qquad i,j = 1,2,3. \tag{3}$$

E_i and B_{ij} are just coefficients depending on the other surrounding charges and on position, while the proportionality to q can be verified experimentally. It is immediate to infer that E_i are the components of the electric field, but the identification of B_{ij} with the magnetic induction (which is of course suggested by the notation) is not so obvious: as a matter of fact, it constitutes the main part of Feynman's notes. Crucial for this passage is to bring relativity into play: the 3-vector in Eq. (3) must be related to the spatial components of a 4-force. By imposing that the Lorentz transformation under a boost of this 4-force is linear in the velocity, Feynman was able both to fix the functional form of the force in (3), thus recovering the Lorentz force, and to derive the behavior of the coefficients E_i and B_{ij} under the boost. And, not surprisingly, such transformation laws turn out to be the correct ones for the electric and magnetic fields.

The reasoning can be sketched as follows. For simplicity we set $q = 1$; it is only a pre-factor that can be reinserted in any moment (i.e. $\mathbf{F} \to q\mathbf{F}$). Also, we set $c = 1$, since it will always be possible to reinsert it by dimensional analysis.

The 4-force associated with the 3-force in (3) is constructed using the well-known formulae from relativistic mechanics, that is:

$$F^\mu = \gamma\,(F_t, F_x, F_y, F_z)\,, \tag{4}$$

where $F_t = \mathbf{F}\cdot\mathbf{v}$ and $\gamma = 1/\sqrt{1-v^2}$. Let us perform a boost with velocity $\mathbf{u}$, for simplicity assumed directed along the x-axis:

$$\left\{ \begin{array}{rcl} x' &=& \gamma_u(x-ut), \\ y' &=& y, \\ z' &=& z, \\ t' &=& \gamma_u(t-ux), \end{array} \right. \qquad \left\{ \begin{array}{rcl} v'_x &=& \dfrac{v_x - u}{1-uv_x}\,, \\[2ex] v'_y &=& \dfrac{v_y}{\gamma_u(1-uv_x)}\,, \\[2ex] v'_z &=& \dfrac{v_z}{\gamma_u(1-uv_x)}\,, \end{array} \right. \tag{5}$$

[5] The fact that the Lorentz force is linear in the velocity was also emphasized by Feynman in chapter 12 of Volume I of his Lectures [2], although there he then derived the complete expression from experiment.

where $\gamma_u = 1/\sqrt{1-u^2}$ is the gamma factor associated with the boost velocity $\mathbf{u}$. Under such a boost, the component x of the 4-vector (4) transforms in the usual way, that is

$$F'_x = \frac{F_x - uF_t}{1 - uv_x}\,. \tag{6}$$

By inserting the explicit expressions for F_x and F_t, and expressing the result in terms of only the primed velocity components, after some algebra[6] we arrive at:

$$\begin{aligned} F'_x \;=\;& E_x - u\gamma_u v'_y E_y - u\gamma_u v'_z E_z + \gamma_u v'_y B_{xy} + \gamma_u v'_z B_{xz} + \frac{v'_x + u}{1+uv'_x} B_{xx} - \frac{u(v'_y)^2 B_{yy}}{1+uv'_x} - \frac{u(v'_z)^2 B_{zz}}{1+uv'_x} \\ & - u\gamma_u \frac{(v'_x+u)v'_y}{1+uv'_x}(B_{xy}+B_{yx}) - \frac{uv'_y v'_z}{1+uv'_x}(B_{yz}+B_{zy}) - u\gamma_u \frac{(v'_x+u)v'_z}{1+uv'_x}(B_{xz}+B_{zx}). \end{aligned} \tag{7}$$

We have now to require that F'_x must have the same form as F_x, that is

$$F'_x = E'_x + v'_x B'_{xx} + v'_y B'_{xy} + v'_z B'_{xz}\,. \tag{8}$$

Since the argument requiring linearity in the velocity should be valid in any inertial frame, this implies that all the terms of (7) that are quadratic in the velocity components should vanish; the same applies to the term in (7) with no component of $\mathbf{v}'$ in the numerator. Then, the B coefficients have to satisfy the following relations:

$$\begin{aligned} B_{xy} + B_{yx} = 0, \qquad B_{yz} + B_{zy} = 0, \qquad B_{xz} + B_{zx} = 0, \\ B_{xx} = 0, \qquad B_{yy} = 0, \qquad B_{zz} = 0. \end{aligned} \tag{9}$$

Such conditions (9) imply that the B coefficients are antisymmetric, i.e. $B_{ij} = -B_{ji}$. It is customary to rearrange the three independent components of an antisymmetric matrix in the form of a 3-component vector, defined in the present case by $B_i = \frac{1}{2}\epsilon_{ijk}B_{jk}$, or $B_x = B_{yz}$, $B_y = B_{zx}$, $B_z = B_{xy}$. This allows us to write the force (3) in the familiar form (where we restore the q dependence)

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \tag{10}$$

which is the usual expression for the Lorentz force acting on the charge q, *provided* we identify the coefficients E and B with the electric and magnetic fields respectively.

2.2 Relativistic invariance and the E, B fields

Such identification requires that those quantities transform correctly under Lorentz boosts, and Feynman's proof proceeds as follows. First of all, by comparing the non-vanishing terms in (7) (after

[6] Useful relations, which may be employed in order to reduce calculations, are the following:

$$1 - uv_x = \frac{\gamma'}{\gamma\gamma_u}\,, \qquad (1+uv_x)(1-uv_x) = 1 - u^2,$$

where $\gamma' = 1/\sqrt{1-v'^2}$.

imposing conditions (9)) with the analogous terms in (8), we can infer the behavior of some of the E, B coefficients under a boost, i.e.

$$E'_x = E_x, \qquad B'_y = \gamma_u(B_y + uE_z), \qquad B'_z = \gamma_u(B_z - uE_y). \tag{11}$$

We recognize here just the usual transformation laws of some of the components of the electric and magnetic field under a boost with velocity $\mathbf{u}$ directed along the positive x-axis: in the present framework, they directly follow from the linearity in the velocity and relativistic mechanics. However, we are not done yet, since the transformation rules for E_y, E_z and B_x still lack. The first two come from requiring that the fourth component of the 4-vector (4) transform in the usual way, i.e. $\gamma' F'_t = \gamma_u \gamma (F_t - uF_x)$. By exploiting the conditions (9) and (11) and the Lorentz transformation laws of the fields we have just derived, as well as the fact that, because of (10), $F_t = \mathbf{E} \cdot \mathbf{v}$ ($q = 1$), along the same lines as above we obtain:

$$\begin{aligned} F'_t &= \frac{1}{1 - v_x}[E_x(v_x - u) + E_y v_y + E_z v_z - u v_y B_z + u v_z B_y] \\ &= v'_x E'_x + v'_y \gamma_u (E_y - uB_z) + v'_z \gamma_u (E_z + uB_y). \end{aligned} \tag{12}$$

And, by imposing that $F'_t = \mathbf{F}' \cdot \mathbf{v}' = \mathbf{E}' \cdot \mathbf{v}'$, the following transformation rules follow

$$E'_y = \gamma_u(E_y - uB_z), \qquad E'_z = \gamma_u(E_z + uB_y), \tag{13}$$

which again are the expected ones. The last transformation rule for B_x can be obtained by considering the third or fourth component of the 4-vector (4). For the boost considered here, such components must be invariant, they being orthogonal to the boost velocity, so that $\gamma' F'_y = \gamma F_y$. By proceeding as above, we get

$$\begin{aligned} F'_y &= \frac{1}{\gamma_u(1 - uv_x)}(E_y - v_x B_z + v_z B_x) \\ &= \gamma_u(1 + uv'_x)E_y - \gamma_u(v'_x + u)B_z + v'_z B_x, \end{aligned} \tag{14}$$

or, by using the inverse transformation laws of E_y and B_z,[7]

$$F'_y = [\gamma_u^2(1 + uv'_x) - \gamma_u^2 v'_x - \gamma_u]E'_y + [\gamma_u^2(1 + uv'_x)u - \gamma_u^2 v'_x - \gamma_u^2 u]B'_z + v'_z B_x. \tag{15}$$

Upon simple manipulation, this reduces to $F'_y = E'_y - v'_x B'_z + v'_z B_x$, so that by imposing this to be of the form $F'_y = E'_y - v'_x B'_z + v'_z B'_x$, we finally get the last transformation law,

$$B'_x = B_x, \tag{16}$$

as expected.

[7] That is: $E_y = \gamma_u(E'_y + uB'_z)$, $B_z = \gamma_u(B'_z + uE'_y)$.

3 Least action principle

A possible way to address the second part of Feynman's point 1 above – i.e. introduce the homogeneous Maxwell equations –can be traced in his Hughes lectures [1], where the vector potential was there introduced in order to write down a relativistically invariant least action principle for a particle in a given potential V. It is conceivable that Feynman had something like this in mind in 1963, when he wrote his notes, but it is as well possible that he thought about this way of proceeding sometime between 1963 and 1967. Part of this discussion is already present in Chapter 19 of Volume II of the Lectures [2], though in a much less detail; similar reasoning is also taken up e.g. by Susskind in his Theoretical Minimum [15].

3.1 Homogeneous Maxwell equations

The starting point is the generalization of the least action principle in classical mechanics to the relativistic case. Since in relativistic mechanics the momentum of a particle of mass m_0 is given by $m_0\mathbf{x}/\sqrt{1-v^2}$, as a first naive guess we may seek for an action S from which the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\mathbf{v}}{\sqrt{1-v^2}}\right] = -\nabla V, \tag{17}$$

can be obtained. Such an action may be identified as

$$S = \int_{t_1}^{t_2}\left[-m_0\sqrt{1-v^2} - V(\mathbf{x},t)\right]\mathrm{d}t = \int_{t_1}^{t_2}\left[-m_0\,\mathrm{d}s - V(\mathbf{x},t)\,\mathrm{d}t\right] \tag{18}$$

($\mathrm{d}s = \sqrt{1-v^2}\,\mathrm{d}t$), since it can be easily checked that, indeed, Eq. (17) just follows from requiring the expression in (18) to be an extremum. The point here, however, is the relativistic invariance of the action or, in Feynman's words, to establish if the action keeps its extremum in any inertial reference frame. Since the potential term is evidently non invariant, it must be modified and, by excluding an invariant scalar potential term $\mathcal{X}(x,y,z,t)\,\mathrm{d}s$ – which does not lead to any known law of nature –, as the next simplest possibility Feynman suggested the use of a 4-potential $A_\mu(x,y,z,t)$, in order for the action to assume the form[8]

$$S = \int_{t_1}^{t_2}\left[-m_0\,\mathrm{d}s - A_t\,\mathrm{d}t + A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_z\,\mathrm{d}z\right] = \int_{t_1}^{t_2}\left[-m_0\,\mathrm{d}s - A_\mu\,\mathrm{d}x^\mu\right]. \tag{19}$$

In the following we shall set $A_t = \phi$, thus matching the usual notation. The next step is, of course, to vary this action in order to see what equations of motion result; this is done in the Appendix, and

[8] Feynman briefly goes on suggesting a 10-potential term of the form $-g_{tt}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s - g_{tx}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}x}{\mathrm{d}s}\mathrm{d}s - \ldots$, mentioning that such term allows to derive the force of gravity. It is interesting to note that in [23] Feynman also gives some references to an approach to gravity which mimics the one given here for electrodynamics, in particular suggesting that in the case of gravity Eq. (3) should be replaced by a quadratic expression.

the result is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\mathbf{v}}{\sqrt{1-v^2}}\right] = \mathbf{F}, \tag{20}$$

with the force **F** given by:

$$\mathbf{F} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}). \tag{21}$$

Such expression looks just like that of the Lorentz force (provided we restore q in front of it, amounting to use $qA_\mu rmdx^\mu$ in place of $A_\mu \mathrm{d}x^\mu$ in the action), when we identify:

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}; \qquad \mathbf{B} = \nabla\times\mathbf{A}\,. \tag{22}$$

Using well-known vector analysis formulae, namely $\nabla\cdot(\nabla\times\mathbf{V}) = 0$ and $\nabla\times(\nabla f)$, valid for any vector field $\mathbf{V}$ and any scalar function f, we can then write down the homogeneous Maxwell equations:

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}. \tag{23}$$

3.2 Gauge transformations

In the Hughes lectures, Feynman deepened his discussion of the least action principle by discussing the gauge transformations of the potentials. Starting again from the action (19), which is now rewritten as

$$S = -m_0\int \mathrm{d}s + \int(\phi - \mathbf{A}\cdot\mathbf{v})\,\mathrm{d}t\,, \tag{24}$$

we can ask whether we could use a different set of potentials ϕ', $\mathbf{A}'$ and still get the same physical trajectory, i.e. the same minimum of the action. This evidently happens if the difference of the action written in terms of the first set of potentials and the action written in terms of the second set of potentials is independent of the path. Such difference,

$$S - S' = \int(\varphi - \mathbf{a}\cdot\mathbf{v})\,\mathrm{d}t\,, \tag{25}$$

($\varphi = \phi - \phi'$, $\mathbf{a} = \mathbf{A} - \mathbf{A}'$) is independent of the integration path if the integrand is a perfect differential, i.e. $\varphi - \mathbf{a}\cdot\mathbf{v} = \mathrm{d}\chi(x,y,z,t)/\mathrm{d}t$, so that

$$\begin{aligned} S - S' &= \int_{t_1}^{t_2}\frac{\mathrm{d}\chi(x,y,z,t)}{\mathrm{d}t} = \int_{t_1}^{t_2}\left(\frac{\partial\chi}{\partial t} + \frac{\partial\chi}{\partial x}\dot{x} + \frac{\partial\chi}{\partial y}\dot{y} + \frac{\partial\chi}{\partial z}\dot{z}\right)\mathrm{d}t \\ &= \chi(x,y,z,t_2) - \chi(x,y,z,t_1)\,. \end{aligned} \tag{26}$$

Comparing (25) with (26) we see that such condition is satisfied if $\varphi = \partial\chi/\partial t$ and $\mathbf{a} = -\nabla\chi$, so that the two actions S and S' lead to the same equations of motion if the potentials are related by the gauge transformations

$$\phi' = \phi + \frac{\partial \chi}{\partial t}, \qquad \mathbf{A}' = \mathbf{A} - \nabla \chi . \tag{27}$$

The fact that the equations of motion do not change under such transformations is also evident from the fact that the electric and magnetic fields defined in (22), which are just the quantities appearing in the equations of motion, are left unchanged. Quite noticeable is this reversed line of reasoning with respect to the standard presentation, which was indeed followed by Feynman, in particular, in his Caltech lectures.

4 Discussion and conclusions

We have discussed a previously unpublished description of electromagnetism developed by Richard Feynman. Motivated by his involvement in undergraduate physics teaching at Caltech in 1961-63, it was obtained only in late 1963, when this involvement was already over, so that it was definitively not published. The approach adopted was similar to the existing ones deriving electromagnetism from special relativity, but manages to go a long way towards the derivation of Maxwell's equations with minimal physical assumptions, in particular without postulating Coulomb's law. To the best of our knowledge, after sketching such an approach, Feynman never used it, even when he taught electromagnetism again, at the Hughes Aircraft Company in 1967. In the latter lectures, however, he adopted an approach which was closer in spirit to the one described here.

As evident from what reported in the previous sections, Feynman's description of electromagnetism was left incomplete, the part remaining in classical electrodynamics discussing the inhomogeneous Maxwell equations: this would require to go well beyond what has been done so far. Indeed, electric and magnetic fields have been considered merely with respect to their effects on charged particles, this allowing to go quite far, up to the homogeneous Maxwell equations and the related gauge transformations, but this is only half of the story. As a matter of fact, it is as well necessary to consider the effect of charges on the fields, i.e. what kind of electric and magnetic fields are generated by charges, and this is evidently the content of inhomogeneous Maxwell's equations. Feynman was of course aware that some new input was necessary at this stage, so that in point 3 of the outline above he proposed to follow the usual experimental route going through Coulomb's inverse square law and Ampère's law for the force between currents. He also suggested that it may be possible to arrive at the wave equation for the potentials (but how?). *De facto*, in the derivation above we have not used any experimental input about the electric and magnetic forces, except the fact that they exist and depend on charge and velocity. In all attempts in the literature to deduce magnetostatics or even full electrodynamics from relativity, one always starts from the experimental Coulomb law.

Bringing to light this approach is of obvious historical significance, adding a further piece to the already variegated Feynman jigsaw, but we believe that such work can have some interest from the point of view of physics teaching, providing a novel way to develop the foundations of electromagnetism, which can fruitfully supplement usual treatments. The above derivation of the Lorentz force, for example, can be considered as an addition to the ordinary undergraduate teaching of electrodynamics, especially pointing out the very clear and compelling logic that allows to determine its linearity in the velocity. The inverse route in the derivation of the homogeneous Maxwell's equation, i.e. first introducing electromagnetic potentials in order to write down a relativistically invariant action, is as well quite interesting for physics teaching. It is then highly desirable that other pieces of Feynman "magic" come to light in the near future. In particular, from [1], [22], and [23], as remarked above, we found evidence that an approach to gravity which is analogous to the one discussed here for electrodynamics was in Feynman's thoughts. The development of such an approach will be the subject of a forthcoming publication [24].

Exercises

1. By using Lorentz's transformations in (5), deduce the following relations:

$$1 + uv'_x = 1 + u\,\frac{v_x - u}{1 - uv_x} = \frac{1 - u^2}{1 - uv_x}\,, \tag{28}$$

$$\sqrt{1 - v'^2} = \frac{\sqrt{1 - v^2}\,\sqrt{1 - u^2}}{1 - uv_x} \tag{29}$$

(or the analogous ones in Footnote 6).

2. Obtain the inverse relations of Eqs. (28),(29), expressing unprimed quantities in terms of primed ones.

3. Write down the explicit expressions for F_x, F_y, F_z in (3) and $F_t = \mathbf{F} \cdot \mathbf{v}$ (put $q = 1$), insert these into Eq. (6) and prove the following expression:

$$\begin{aligned} F'_x \;=\; & \frac{1}{1 - uv_x}[E_x(1 - uv_x) - uv_y E_y - uv_z E_z + v_x(1 - uv_x)B_{xx} + v_y B_{xy} + v_z B_{xz} \\ & - uv_x v_y(B_{xy} + B_{yx}) - uv_z v_x(B_{xz} + B_{zx}) - uv_y^2 B_{yy} - uv_z^2 B_{zz} - uv_x v_z(B_{yz} + B_{zy})]\,. \end{aligned} \tag{30}$$

From this, using the results in the previous exercises, deduce Eq. (7).

4. Derive the equations of motion (17) from the action (18) along the lines of the Appendix.

Appendix A Variational calculations

By following [1], we here find the Euler-Lagrange equations for the action (19), which we rewrite here as follows:

$$S = -\int_{t_i}^{t_f} \left(-m_0\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2} - \phi + A_x\dot{x} + A_y\dot{y} + A_z\dot{z}\right) \mathrm{d}t \tag{A.1}$$

We have to vary the path $(x(t), y(t), z(t))$ by infinitesimal quantities $(\xi(t), \eta(t), \zeta(t))$,

$$x(t) \to x(t) + \xi(t), \quad y(t) \to y(t) + \eta(t), \quad z(t) \to z(t) + \zeta(t), \tag{A.2}$$

with the constraint that the variations vanish at the endpoints t_i, t_f, and keep only first order terms in ξ, η and ζ and their time derivatives. This induces the action S to vary as well by $S \to S + \delta S$ and, by requiring it to be stationary, the equations of motion for all three coordinates x, y and z follow. This procedure, however, would involve a lot of terms, so that, by following Feynman, we choose to vary only one coordinate at a time, in order to get only one component of the equation of motion: since calculations are very similar in the three cases, we here perform it only for the x case, leaving the other two to the reader. We have:

$$\begin{aligned} S + \delta S \;=\; & \int_{t_i}^{t_f} \Big[-m_0\sqrt{1-(\dot{x}+\dot{\xi})^2-\dot{y}^2-\dot{z}^2} \;-\; \phi(x+\xi, y, z, t) \;+\; A_x(x+\xi, y, z)(\dot{x}+\dot{\xi}) \\ & + A_y(x+\xi, y, z, t)\dot{y} \;+\; A_z(x+\xi, y, z, t)\dot{z}\Big] \mathrm{d}t. \end{aligned} \tag{A.3}$$

By Taylor expanding at first order the integrand, we have:

$$\begin{aligned} \sqrt{1-(\dot{x}+\dot{\xi})^2-\dot{y}^2-\dot{z}^2} \;&\simeq\; \sqrt{1-v^2-2\dot{x}\dot{\xi}} \simeq \sqrt{1-v^2}\left(1 - -\frac{\dot{x}\dot{\xi}}{1-v^2}\right) = \sqrt{1-v^2} - \frac{\dot{x}\dot{\xi}}{\sqrt{1-v^2}}, \\ \phi(x+\xi, y, z, t) \;&\simeq\; \phi(x, y, z, t) \;+\; \xi\,\frac{\partial\phi}{\partial x}(x, y, z, t), \\ A_x(x+\xi, y, z, t) \;&\simeq\; A_x(x, y, z, t) \;+\; \xi\,\frac{\partial A_x}{\partial x}(x, y, z, t), \\ A_y(x+\xi, y, z, t) \;&\simeq\; A_y(x, y, z, t) \;+\; \xi\,\frac{\partial A_y}{\partial x}(x, y, z, t), \\ A_z(x+\xi, y, z, t) \;&\simeq\; A_z(x, y, z, t) \;+\; \xi\,\frac{\partial A_z}{\partial x}(x, y, z, t), \end{aligned}$$

so that:

$$\begin{aligned} S + \delta S \;\simeq\; & \int_{t_i}^{t_f} \left(-m_0\sqrt{1-v^2} - \phi + A_x\dot{x} + A_y\dot{y} + A_z\dot{z}\right) \mathrm{d}t \\ & + \int_{t_i}^{t_f} \left(\frac{m_0\dot{x}}{\sqrt{1-v^2}} + A_x\right)\dot{\xi} \;+\; \int_{t_i}^{t_f} \xi\left(-\frac{\partial\phi}{\partial x} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right) \mathrm{d}t. \end{aligned} \tag{A.4}$$

The first integral build up again the action S we started with, while the second one can be integrated by parts to give:

$$\int_{t_i}^{t_f}\left(\frac{m_0\dot{x}}{\sqrt{1-v^2}}+A_x\right)\dot{\xi}\,\mathrm{d}t=\left[\left(\frac{m_0\dot{x}}{\sqrt{1-v^2}}+A_x\right)\xi\right]_{t_i}^{t_f}-\int_{t_i}^{t_f}\xi\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\dot{x}}{\sqrt{1-v^2}}+A_x\right)\mathrm{d}t, \tag{A.5}$$

where the first term vanishes since we assumed that at the endpoints $\xi(t_i)=\xi(t_f)=0$. Therefore we get:

$$\delta S=\int\xi\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0\dot{x}}{\sqrt{1-v^2}}-\dot{A}_x-\frac{\partial\phi}{\partial x}+\frac{\partial A_x}{\partial x}\,\dot{x}+\frac{\partial A_y}{\partial x}\,\dot{y}+\frac{\partial A_z}{\partial x}\,\dot{z}\right)\mathrm{d}t. \tag{A.6}$$

Here, $\dot{A}_x$ is a total derivative, i.e.

$$\dot{A}_x=\frac{\partial A_x}{\partial x}\dot{x}+\frac{\partial A_x}{\partial y}\dot{y}+\frac{\partial A_x}{\partial z}\dot{z}+\frac{\partial A_x}{\partial t}\,. \tag{A.7}$$

By requiring $\delta S=0$, being the variation ξ arbitrary, the equation of motion follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\dot{x}}{\sqrt{1-v^2}}\right]=-\frac{\partial\phi}{\partial x}-\frac{\partial A_x}{\partial t}+\dot{y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+\dot{z}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right). \tag{A.8}$$

Similarly, by requiring the action to be stationary when varying $y(t)$ and $z(t)$, we arrive at the other two equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\dot{y}}{\sqrt{1-v^2}}\right]=-\frac{\partial\phi}{\partial y}-\frac{\partial A_y}{\partial t}+\dot{z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\dot{x}\left(\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}\right), \tag{A.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\dot{z}}{\sqrt{1-v^2}}\right]=-\frac{\partial\phi}{\partial z}-\frac{\partial A_z}{\partial t}+\dot{x}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\dot{y}\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right). \tag{A.10}$$

In vector form, these three equations of motion rewrite as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\mathbf{v}}{\sqrt{1-v^2}}\right]=\mathbf{F} \tag{A.11}$$

with $\mathbf{F}=-\nabla\phi-\dfrac{\partial}{\partial t}\mathbf{A}+\mathbf{v}\times(\nabla\times\mathbf{A})$, as in Eq. (21).

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