Ma432 Classical Field Theory

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These notes cover a lot of the 2008-2009 Ma432 Classical Field Theory course given by Dr Nigel Buttimore (replaced by Ma3431 Classical Field Theory and Ma3432 Classical Electrodynamics, the former corresponding to at least the first four sections of these notes). The emphasis is mostly on the Lagrangian formulation of classical electrodynamics and the solution of Maxwell's equations by Green's function methods. They are probably slightly suspect, particularly with regard to indices and brackets (and no doubt contain other more unsettling errors).
I am told that Dr Buttimore has changed his units from those in these notes, so use at your own discretion.

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1.1 Introduction to field theory

You are probably already familiar with the notion of electric and magnetic fields. Loosely speaking, a field in a physics is a physical quantity defined at every point of space and time, which can be valued as a single number (scalar field), a vector (vector field, such as electromagnetism and gravity), or as a tensor.

We will restrict ourselves to the study of the electric and magnetic fields, and do so in a unified manner consistent with special relativity. The methods we use are based in the Lagrangian approach to classical mechanics. Recall that in classical mechanics the Lagrangian L was defined as $L = T - V$ where T and V are the kinetic and potential energies of the system in question. The action of the system was defined to be the quantity $S = \int L dt$, and the equations of motion of the system were found from the principle of least action, which states that the true time evolution of the system is such that the action is an extremum. The equations of motion (known as the Euler-Lagrange equations) were thus derived from the condition $\delta S = \delta \int L dt = 0$.

In studying fields which take on different values at different space points it is convenient to express the Lagrangian itself as an integral, $L = \int d^3x\, \mathcal{L}$, where $\mathcal{L}$ is called the Lagrangian density. The full action is then $S = \int dt d^3x\, \mathcal{L}$. Note that when we approach this from the special relativistic point of view the separate time and space components will be unified into a single package.

1.2 Field theory as a continuum limit

To begin, let us show how a simple field theory may be derived by taking the continuum limit of a system of N particles on a spring with spring constant k. Let the particles have equilibrium positions $a, 2a, \ldots Na$ and denote the deviation of the i^{th} particle from its equilibrium by ϕ_i. The force on the i^{th} particle is

$$F_i = \begin{cases} +k(\phi_2 - \phi_1) & i = 1 \\ -k(\phi_i - \phi_{i-1}) + k(\phi_{i+1} - \phi_i) & 1 < i < N \\ -k(\phi_N - \phi_{N-1}) & i = N \end{cases}$$

The Lagrangian is

$$L = T - V = \sum_{i=1}^{N} \frac{1}{2} m \dot{\phi}_i^2 - \sum_{i=1}^{N} \frac{1}{2} k (\phi_{i+1} - \phi_i)^2$$

and the equations of motion are

$$m\ddot{\phi}_i = -k(\phi_i - \phi_{i-1}) + k(\phi_{i-1} - \phi_i) \qquad 1 < i < N$$

We now take the limit $a \to 0$ while keeping $(N-1)a$ fixed by letting $N \to \infty$. If we write $x = ai$ as the position of the i^{th} particle then we can regard $\phi_i \equiv \phi(x = ai, t)$, and using the equations of motion in the form

$$\frac{m}{a}\ddot{\phi}_i = ka\frac{1}{a^2}\Big[(\phi_{i-1} - \phi_i) - (\phi_i - \phi_{i-1})\Big]$$

we can apply the definition of the derivative

$$\left.\frac{\partial \phi}{\partial x}\right|_i = \lim_{a\to 0}\frac{\phi\left([i+1]a\right) - \phi(ai)}{a}$$

twice to obtain the equations of motion in the limit $a \to 0$:

$$\mu\frac{\partial^2 \phi}{\partial t} = \kappa\frac{\partial^2 \phi}{\partial x^2}$$

where $\kappa = \lim_{a\to 0} ka$ and $\mu = \frac{m}{a}$ is the mass density which we keep fixed. We see that our simple field obeys the wave equation.

If we define

$$L = \sum_{i=1}^{N} a\mathcal{L}_i \qquad \mathcal{L}_i = \frac{1}{2}\mu\dot{\phi}_i^2 - \frac{1}{2}\frac{k}{a}\left(\phi_{i+1} - \phi_i\right)^2$$

then in the limit we obtain the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\kappa\left(\frac{\partial \phi}{\partial x}\right)^2$$

such that $L = \int dx\mathcal{L}$.

1.3 Euler-Lagrange equations

A more general Lagrangian density would be of the form $\mathcal{L}(\partial_t\phi_i, \partial_x\phi_i, \phi_i, t, x)$. We can use Hamilton's Principle of Least Action to find the general form of the equations of motion.

Let us consider the simplest case where the field is one-dimensional and the Lagrangian density is invariant under time and space translation. Then we have that

$$\int \delta\mathcal{L}\, dx\, dt = 0$$

and in full

$$\int\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta(\partial_t\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\delta(\partial_x\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi\right) dx\, dt = 0$$

Noting that $\delta(\partial_t\phi) = \partial_t(\delta\phi)$ and $\delta(\partial_t\phi) = \partial_t(\delta\phi)$ we rewrite this as

$$\int\left(\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta\phi\right) - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta\phi + \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\delta\phi\right) - \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi\right) dx\, dt = 0$$

We integrate out the total time and space derivatives and use the fact that the $\delta\phi$ term must vanish at the endpoints to then obtain

$$\delta S = \int \left(\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi \, dx \, dt = 0$$

hence we obtain the Euler-Lagrange equations for this field:

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

For a vector field just replace ϕ by A^i.

2 Special relativity

We will now introduce the machinery that allows us to express field theory in a manner consistent with the theory of special relativity. In particular, we seek to formulate the theory of fields in a manner that is Lorentz covariant - that is, related from one frame to another via Lorentz transformations. Note that we do not introduce special relativity systematically but assume some prior knowledge of the subject. For completeness we note that the two postulates of special relativity are that the laws of physics take the same form in all inertial (non-accelerating) reference frames, and that the speed of light c in vacuum is an absolute constant regardless of frame.

2.1 Rapidity

The basic Lorentz transformations in $1+1$ dimensions are

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad x' = \gamma(x - vt) \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

for a frame S' moving with velocity v with respect to the frame S. We have that

$$ct' - x' = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}(ct - x)$$

We define the rapidity $\zeta(v)$ as

$$\zeta(v) = \frac{1}{2}\ln\left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)$$

so that

$$ct' - x' = e^{\zeta}(ct - x)$$

Note that rapidities add. We can then show that

$$v = \tanh\zeta \qquad \gamma = \cosh\zeta$$

which allows us to write the Lorentz transformations as

$$ct' = ct\cosh\zeta - x\sinh\zeta \qquad x' = x\cosh\zeta - ct\sinh\zeta$$

In the full $1+3$ dimensions we can write this transformation in matrix form as

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

called a boost in the x-direction. Note that the most general proper Lorentz transformation can be written as a product of a 3-rotation to align the new x-axis with the direction of motion, a boost along the new x-direction with velocity v and a second 3-space rotation.

2.2 Tensor notation

A basic invariant in special relativity is the interval ds separating two (infinitesimally close) events in four-dimensional space-time: $ds^2 = c^2dt^2 - dx_1^2 - dx_2^2 - dx_3^2$. From this we get the metric tensor:

$$g^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1) = g_{\mu\nu}$$

which is used to raise and lower indices as follows:

$$x^\mu = g^{\mu\alpha} x_\alpha \qquad x_\mu = g_{\mu\alpha} x^\alpha$$

Note that we sum over repeated indices. Upper indices are said to be contravariant, and lower indices are said to be covariant. Note that (in this metric) raising a time-index has no effect, $x_0 = x^0$, while raising a space-index changes the sign, $x_i = -x^i$. Note also that indices with Greek letters can take any value in $\{0, 1, 2, 3\}$ while indices with Roman letters refer to spatial indices, $\{1, 2, 3\}$. Thus in our notation we have $x^\mu = (ct, \vec{x})$ and $x^i = \vec{x}$.

A Lorentz transformation relates events x' in the frame S' to events x in the frame S, and is written as

$$x'^{\,\mu} = \Lambda^\mu_\beta x^\beta$$

The metric tensor is invariant under Lorentz transformations

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta$$

A four-dimensional vector, or four-vector, is written as A^μ and transforms like the coordinates x^μ:

$$A'^\mu = \Lambda^\mu_\nu A^\nu$$

while a second rank tensor $T^{\mu\nu}$ transforms like the product of (components of) two four-vectors:

$$T'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$$

and similarly for tensors of higher rank.

We form the four-dimensional Kronecker delta by lowering the index of $g^{\mu\nu}$:

$$g_{\lambda\nu} g^{\nu\mu} = g^\mu_\lambda = \delta^\mu_\lambda = \mathrm{diag}\,(1, 1, 1, 1)$$

We also have the four-dimensional Levi-Civita symbol, $\epsilon^{\alpha\beta\gamma\delta}$ which is $+1$ for even permutations of $\alpha\beta\gamma\delta$ and -1 for odd permutations, with $\epsilon^{0123} = 1$. Note that for the covariant form, $\epsilon_{\alpha\beta\gamma\delta}$, we have $\epsilon_{0123} = -1$. Both the Levi-Civita symbol and the Kronecker delta are invariant under Lorentz transformations.

We can form a scalar invariant under Lorentz transformations (a Lorentz scalar) by contracting two four vectors

$$a^\mu b_\mu = a'^\mu b'_\mu$$

The invariant time element $d\tau$ is given by

$$c^2 d\tau^2 = dx^\mu dx_\mu$$

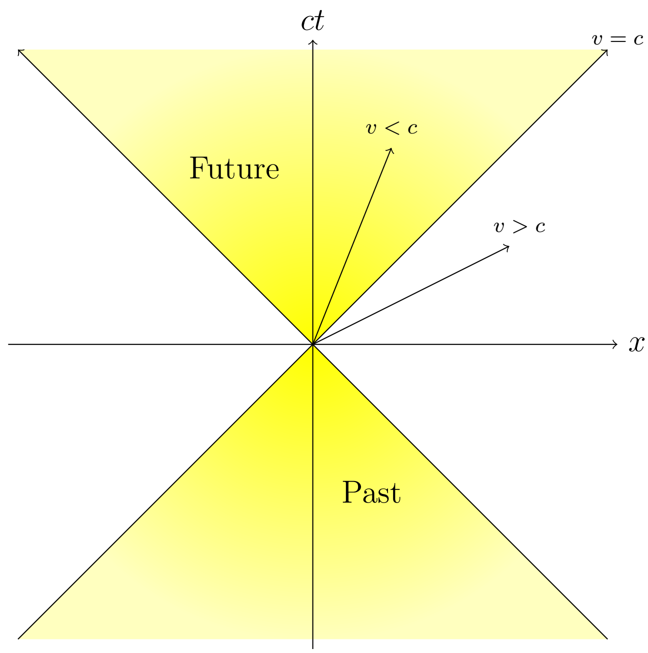
It is related to the usual time element by

$$d\tau = \frac{dt}{\gamma} \Rightarrow \frac{d}{d\tau} = \gamma \frac{d}{dt}$$

This allows us to define vectors of four-velocity $V^\mu = \frac{dx^\mu}{d\tau}$ and four-momentum $p^\mu = m\frac{dx^\mu}{d\tau}$ where m is the rest mass of the particle. The zero component of the four-momentum is related to the energy $\mathcal{E} = \gamma mc^2$ by $cp_0 = \mathcal{E}$, so we can write the four-momentum as $p^\mu = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$. Note that the contraction of the four-momentum with itself is $p^\mu p_\mu = m^2c^2$.

If this (or indeed the scalar formed by contracting any four-vector with itself) is equal to zero we say that the four-vector is light-like, if it is greater than zero we say that it is space-like, if it is less than zero it is timelike.

This can be related to the idea of light-cones:



This picture can be understood as follows: events that occur inside the lightcone are timelike; it is possible to find a Lorentz transformation such that any two events occur at the same point in space, but at different times. Similarly, events that occur outside the lightcone are spacelike in that is possible to find a Lorentz transformation to a frame such that any two events occur at the same point in time, but are separated in space.

The boundary of the light-cone is given by a line corresponding to the motion of a particle of velocity c. The motion of a particle with velocity less than c lies within its light-cone.

Events that occur within the past light-cone of a particle can affect the particle in the present, while events that occur outside it cannot.

Finally we must consider calculus in space-time. We will be integrating over the four-dimensional volume element

$$d^4x = dx_0 dx_1 dx_2 dx_3 = c\,dt\,dx_1 dx_2 dx_3$$

which is an invariant. Derivatives are denoted by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

Note that differentiating with respect to a lower index gives an upper index, while differentiating with respect to an upper index gives a lower index, so for instance

$$\frac{\partial x_\mu}{\partial x_\nu} = \delta_\mu^\nu \qquad \frac{\partial x^\mu}{\partial x_\nu} = g^{\mu\rho}\frac{\partial x_\rho}{\partial x_\nu} = g^{\mu\rho}\delta_\rho^\nu = g^{\mu\nu} \qquad \frac{\partial(\partial_\alpha x_\beta)}{\partial(\partial_\mu x_\nu)} = \delta_\alpha^\mu \delta_\beta^\nu$$

3 Covariant field theory

We now seek to formulate field theories in a special relativistic context. We will be seeking actions

$$S = \int \mathcal{L}\, d^3x\, dt = \int L\, dt = \int L\gamma\, d\tau$$

which are relativistically invariant. Recall that the first postulate of special relativity is that the laws of physics are the same in all (inertial) reference frames - these laws take the form of the equations of motion derived from the condition $\delta S = 0$, hence S must be Lorentz invariant. As $d\tau$ is an invariant scalar we see that we must have $L\gamma$ also a Lorentz scalar. The simplest way to achieve this is to contract the available four-vectors.

Note that the Euler-Lagrange equations of motion for a field A^μ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

3.1 Relativistic free particle action

For a free particle, the only scalar which preserves translational invariance is $p_\mu p^\mu = (mc)^2$, suggesting a Lagrangian of the form $\mathcal{L} = Cp_\mu p^\mu = Cm^2c^2$, where C is some constant. Let us look at the non-relativistic limit of this Lagrangian. We have

$$S = \int L\, dt = \int L\gamma\, d\tau$$

We want the non-relativistic limit of L to agree with the Lagrangian for a non-relativistic free particle, $L = \frac{1}{2}m\vec{v}^2$. Consider the Taylor expansion of $m\gamma^{-1}$:

$$m\gamma^{-1} = m\left(1 - \frac{\vec{v}^2}{c^2}\right) = m - \frac{m}{2}\frac{\vec{v}^2}{c^2} + O\left(\frac{1}{c^4}\right)$$

$$\Rightarrow -mc^2\gamma^{-1} = -mc^2 + \frac{1}{2}m\vec{v}^2$$

As the constant term $-mc^2$ is unimportant, we see that we can take our Lagrangian to be

$$L = -\frac{mc^2}{\gamma}$$

hence we have action

$$S = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int d\tau$$

or

$$S = -\frac{1}{m} \int p_\mu p^\mu d\tau$$

3.2 Relativistic interactions

We now let our particle be acted on by some field with potential $A_\mu(x^\nu)$. Possible scalars include $A_\mu p^\mu$ and $A_\mu A^\mu$. Again we want to choose the Lagrangian so that the non-relativistic limit gives us the interaction Lagrangian for a particle in the presence of an electric field (in the non-relativistic limit the particle's velocity goes to zero, and so it will not interact with the magnetic field). This is $L = -e\Phi$. This suggests we look at $eA_\mu p^\mu = ep^0 A_0 - e\vec{v}\cdot\vec{A}$. Now, $p_0 = \frac{\mathcal{E}}{c} = m\gamma c$, and we identify A^0 with Φ. Then in the limit $\vec{v} \to 0$ we have $ep_\mu A^\mu = mc\gamma e\Phi$. Thus we take our interaction term to be

$$L_{int} = -\frac{e}{mc\gamma} A_\mu p^\mu$$

The total action is now

$$S = -\frac{1}{m}\int \left(p_\mu + \frac{e}{c}A_\mu\right) p^\mu d\tau = -\int \left(p_\mu + \frac{e}{c}A_\mu\right) dx^\mu$$

using $p^\mu = m\frac{dx^\mu}{d\tau}$. The dynamics of the system can then be found by varying this action.

First, let us note that

$$\delta(p_\mu p^\mu) = \delta p_\mu p^\mu + p_\mu \delta p^\mu = p_\mu \delta p^\mu + p_\mu \delta p^\mu = 2p_\mu \delta p^\mu = 2m\frac{dx_\mu}{d\tau}\delta p^\mu$$

but also

$$\delta(p_\mu p^\mu) = \delta(m^2c^2) = 0$$

and hence if $m \neq 0$ we have

$$\frac{dx_\mu}{d\tau}\delta p^\mu = 0 \Rightarrow dx_\mu \delta p^\mu = 0$$

Now let us compute the variation:

$$\begin{aligned}
\delta S &= -\delta \int \left(p_\mu + \frac{e}{c}A_\mu\right) dx^\mu \\
&= -\int \left(p_\mu + \frac{e}{c}A_\mu\right) \delta(dx^\mu) - \int \left(\delta p_\mu + \frac{e}{c}\delta A_\mu\right) dx^\mu \\
&= -\int \left(p_\mu + \frac{e}{c}A_\mu\right) d(\delta x^\mu) - \frac{e}{c}\int \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu dx^\mu
\end{aligned}$$

where we have used the above result to eliminate the δp_μ term. We now use

$$\begin{aligned}
\int \left(p_\mu + \frac{e}{c}A_\mu\right) d(\delta x^\mu) &= \int d\left(p_\mu + \frac{e}{c}A_\mu(\delta x^\mu)\right) - \int d\left(p_\mu + \frac{e}{c}A_\mu\right)\delta x^\mu \\
&= \int \frac{d}{d\tau}\left(p_\mu + \frac{e}{c}A_\mu(\delta x^\mu)\right) d\tau - \int d\left(p_\mu + \frac{e}{c}A_\mu\right)\delta x^\mu
\end{aligned}$$

and the fact that the variation δx^μ vanishes at the end points to obtain

$$\begin{aligned}\delta S &= -\underbrace{\left[\left(p_\mu + \frac{e}{c}A_\mu\right)\delta x^\mu\right]_{\tau_1}^{\tau_2}}_{=0} + \int \left(dp_\mu + \frac{e}{c}dA_\mu\right)\delta x^\mu - \frac{e}{c}\int \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu dx^\mu \\ &= \int \left(dp_\mu + \frac{e}{c}\frac{\partial A_\mu}{\partial x^\nu}dx^\nu\right)\delta x^\mu - \frac{e}{c}\int \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu dx^\mu \\ &= \int \left[\frac{dp_\mu}{d\tau}d\tau + \frac{e}{c}\left(\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\delta x^\mu - \frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right)d\tau\right]\end{aligned}$$

Switching the dummy variables μ and ν in the second A_μ term we have:

$$\delta S = \int d\tau \delta x^\mu \left[\frac{dp_\mu}{d\tau} + \frac{e}{c}\left(\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} - \frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{d\tau}\right)\right] = 0$$

Hence we find equations of motion

$$\frac{dp_\mu}{d\tau} = \frac{e}{c}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)\frac{dx^\nu}{d\tau}$$

3.3 Electromagnetic field tensor

The electric and magnetic fields can be expressed in terms of the 4-potential A^μ as

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\Phi \quad (A_0 \equiv \Phi)$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Note in passing that $\vec{E}$ has odd parity (i.e. transforms as $\vec{E} \mapsto -\vec{E}$ under space reversal, $\vec{r} \mapsto -\vec{r}$) and is even under time reversal, while $\vec{B}$ has even parity and is odd under time reversal.

We now introduce the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where we identify

$$F^{i0} = E^i \qquad F^{ij} = -\varepsilon^{ijk}B^k$$

as we have

$$E^i = -\partial_0 A^i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = F^{i0}$$

$$-\varepsilon^{ijk}B^k = -\varepsilon^{ijk}\varepsilon_{klm}\partial_l A^m = -(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l)\partial_l A^m = -\partial_i A^j + \partial_j A^i = \partial^i A^j - \partial^j A^i = F^{ij}$$

Explicitly, the contravariant and covariant forms of the tensor are:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

The equations of motion derived in the last section can then be written

$$\frac{dp_\mu}{d\tau} = \frac{e}{c} F_{\mu\nu} \frac{dx^\nu}{d\tau}$$

To write these in three-dimensional form we first set $\mu = 0$ and sum over ν (noting that $\frac{d}{d\tau} = \gamma \frac{d}{dt}$)

$$\frac{dp_0}{dt} = \frac{e}{c} F_{0i} \frac{dx^i}{dt} = \frac{e}{c} F^{i0} v^i \Rightarrow \frac{d\mathcal{E}}{dt} = e\vec{E} \cdot \vec{v}$$

using $p_0 = \frac{\mathcal{E}}{c}$ and the fact that raising a spatial index changes the sign, as well as the antisymmetry of $F_{\mu\nu}$. For $\mu = i$

$$\frac{dp_i}{dt} = \frac{e}{c} F_{i0} \frac{dx^0}{dt} + \frac{e}{c} F_{ij} \frac{dx^j}{dt} \Rightarrow -\frac{dp^i}{dt} = -eF^{i0} - \frac{e}{c} \varepsilon_{ijk} v^k B^k$$

$$\Rightarrow \frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c} \vec{v} \times \vec{B}$$

Note that $F_{\mu\nu}$ is invariant under an important class of transformations known as **gauge transformations**. A gauge transformation of the electromagnetic four-potential is a transformation of the form $A_\mu \mapsto A_\mu + \partial_\mu \phi$ for some scalar field ϕ, and under this we have

$$F_{\mu\nu} \mapsto \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \phi - \partial_\nu \partial_\mu \phi = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

The invariance of the electromagnetic field tensor and hence the observable fields allows us to simplify problems by choosing a particular gauge, i.e. a particular choice of the A_μ satisfying certain conditions. For example we will later explicitly solve Maxwell's equations (introduced in the next section) in Lorenz gauge: $\partial_\mu A^\mu = 0$.

3.4 Maxwell's equations

In the moonlight opposite me were three young women, ladies by their dress and manner. I thought at the time that I must be dreaming when I saw them, they threw no shadow on the floor.

Bram Stoker, *Dracula*

The simplest choice of a Lagrangian density for the electromagnetic field tensor is $\mathcal{L} = CF_{\mu\nu}F^{\mu\nu}$ where C is some constant. We will now find the equations of motion satisfied by the field using the Euler-Lagrange equations. We have

$$\begin{aligned}\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} &= 2F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \\ &= 2F^{\alpha\beta}\left(\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} - \frac{\partial(\partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)}\right) \\ &= 2F^{\alpha\beta}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right) \\ &= 2F^{\mu\nu} - 2F^{\nu\mu} \\ &= 4F^{\mu\nu}\end{aligned}$$

and as $\frac{\partial\mathcal{L}}{\partial A_\nu} = 0$ we have the equation of motion for a free field

$$\partial_\mu F^{\mu\nu} = 0$$

If we introduce a source of charge and 3-current $J^\mu = (c\rho, \vec{J})$ (see the next section) then we have an interaction term

$$\mathcal{L}_{int} = -\frac{1}{c}A_\mu J^\mu$$

and we obtain

$$\partial_\mu F^{\mu\nu} = -\frac{1}{4Cc}J^\nu$$

and choosing $C = -\frac{1}{16\pi}$ (i.e. Gaussian cgs units) gives

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$$

In three dimensions this becomes

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

and

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$$

These are two of **Maxwell's equations**. The other two may be expressed using the dual tensor $\tilde{F}^{\mu\nu}$ defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\sigma\rho} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma$$

The components of the dual tensor may be seen to be

$$\tilde{F}^{i0} = B^i \qquad \tilde{F}^{ij} = \epsilon^{ijk}E^k$$

in words, replace $\vec{E}$ with $\vec{B}$ and $\vec{B}$ with $-\vec{E}$. Now consider

$$\partial_\mu\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma$$

This is the contraction of an antisymmetric tensor with a symmetric tensor, and so is equal to zero. Hence we have the equation

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

By comparing with the above Maxwell's equations in the case that $J^\mu = 0$ and interchanging the electric and magnetic fields in the manner mentioned above, we find the other two Maxwell's equations:

$$\vec{\nabla} \cdot \vec{B} = 0$$

and

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \vec{\nabla} \times \vec{E} = 0$$

Note that the equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$ can also be written in the form

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

The various contractions arising from the field tensor and its dual are

$$F_{\mu\nu}F^{\mu\nu} = 2\left(\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}\right) \qquad F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B} \qquad \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = 2(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B})$$

3.5 Four-current and charge conservation

The four-current density is given by

$$J^\mu(t,\vec{x}) = e\frac{dx^\mu}{dt}\delta^3(\vec{x} - \vec{x}_e(t))$$

where $\vec{x}_e(t)$ is the path of a particle of charge e generating the field A_μ. The charge q and current $e\vec{v}$ are then given by

$$q = \int \rho(t,\vec{x})d^3x \qquad e\vec{v} = \int \vec{J}(t,\vec{x})d^3x$$

Consider the gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu\phi$. This gives an interaction action

$$S' = -\frac{1}{c^2}\int \left(J^\mu A_\mu + J^\mu\partial_\mu\phi\right) d^4x$$

We vary this with respect to ϕ:

$$\delta S' = -\frac{1}{c^2}\int \delta(J^\mu\partial_\mu\phi)d^4x = -\frac{1}{c^2}\int \left(-\partial_\mu J^\mu\delta\phi + \partial_\mu[J^\mu\delta\phi]\right) d^4x$$

The second term vanishes on the boundary, and we are left with

$$\delta S' = \frac{1}{c^2}\int \partial_\mu J^\mu \delta\phi d^4x = 0 \Rightarrow \partial_\mu J^\mu = 0$$

so charge is conserved. This equation can also be written

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Let us integrate this over a surface Σ,

$$\int_\Sigma \frac{\partial \rho}{\partial t} d^3x + \int_\Sigma \vec{\nabla} \cdot \vec{J} d^3x = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \int_\Sigma \rho \, d^3x + \int_{\partial\Sigma} \vec{J} \cdot d\vec{A} d^3x = 0$$

or

$$\frac{\partial q}{\partial t} + \int_{\partial\Sigma} \vec{J} \cdot d\vec{A} = 0$$

We see that invariance under gauge transformations leads to charge conservation. In fact there is a close relationship between certain types of transformational invariance and conservation laws, which we treat in detail in the next section.

4 Noether's theorem

Noether's theorem states for every continuous symmetry there is a conserved quantity.

4.1 Derivation

Let us suppose we have a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi, \varepsilon)$ invariant under a transformation

$$\phi \to \phi^* = \phi + \delta\phi \qquad x^\mu \to x^{*\mu} = x^\mu + \Delta x^\mu$$

parametrised by a small quantity ε such that

$$\phi^*(\varepsilon = 0) = \phi \qquad x^{*\mu}(\varepsilon = 0) = x^\mu$$

Now the change in x^μ means that the volume we integrate over will change; that is, we have

$$c\delta S = \int_{\Sigma'} \mathcal{L}(\phi^*, x^{*\mu}) d^4x^* - \int_{\Sigma} \mathcal{L}(\phi, x^\mu) d^4x$$

$$\Rightarrow c\delta S = \int_{\Sigma} [\mathcal{L}(\phi^*, x^\mu) - \mathcal{L}(\phi, x^\mu)] d^4x + \int_{\partial\Sigma} \mathcal{L} \Delta x^\mu d\Sigma_\mu$$

where $d\Sigma_\mu$ is a surface element (in the x_μ-direction) and $\Delta x^\mu d\Sigma_\mu$ can be thought of as giving the change in the boundary $\partial\Sigma$ caused by the transformation of the coordinates (for more details see *Classical Mechanics* by Goldstein, 3rd edition, page 592). We have also switched dummy variables in the first integral from $x^{*\mu}$ to x^μ.

Now the first integrand is just the variation of $\mathcal{L}$ with respect to ϕ, that is

$$\int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^4x = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{=0 \text{ (E-L)}} \delta\phi \right] d^4x$$

while the second can be rewritten using the divergence theorem

$$\int_{\partial\Sigma} \mathcal{L} \Delta x^\mu d\Sigma_\mu = \int_{\Sigma} \partial_\mu \left(\mathcal{L} \Delta x^\mu \right) d^4x$$

hence

$$c\delta S = \int_{\Sigma} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \Delta x^\mu \right] d^4x = 0$$

Now, consider

$$\begin{aligned} \delta\phi &= \phi^*(x^\rho) - \phi(x^\rho) \\ &= \phi^*(x^{*\rho}) - \phi(x^\rho) - [\phi^*(x^{*\rho}) - \phi^*(x^\rho)] \end{aligned}$$

We now Taylor expand $\phi^*(x^{*\rho})$ with respect to x^ρ, obtaining

$$\phi^*(x^{*\rho}) - \phi^*(x^\rho) \approx \phi^*(x^\rho) + \partial_\nu \phi^*(x^\rho)\Delta x^\nu - \phi^*(x^\rho) = \partial_\nu \phi^*(x^\rho)\Delta x^\nu$$

or, using $\phi^* = \phi + \delta\phi$

$$\phi^*(x^{*\rho}) - \phi^*(x^\rho) \approx \partial_\nu \phi(x^\rho)\Delta x^\nu + \partial_\nu \delta\phi \Delta x^\nu$$

and the term on the right is of second order and so can be neglected.

Similarly we expand $\phi^*(x^\rho)$ with respect to ε to find

$$\phi^*(x^{*\rho}) - \phi(x^\rho) \approx \phi^*(x^{*\rho})\Big|_{\varepsilon=0} + \varepsilon \frac{\partial \phi^*}{\partial \varepsilon}\bigg|_{\varepsilon=0} - \phi(x^\rho) = \phi(x^\rho) + \varepsilon \frac{\partial \phi^*}{\partial \varepsilon}\bigg|_{\varepsilon=0} - \phi(x^\rho) = \varepsilon \frac{\partial \phi^*}{\partial \varepsilon}\bigg|_{\varepsilon=0}$$

and as we also have $\Delta x^\mu = x^{*\mu} - x^\mu \approx \frac{\partial x^{*\mu}}{\partial \varepsilon}|_{\varepsilon=0}$,

$$c\delta S = \int_\Sigma \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \varepsilon \frac{\partial \phi^*}{\partial \epsilon}\bigg|_{\epsilon=0} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \varepsilon \frac{\partial x^{*\nu}}{\partial \epsilon}\bigg|_{\epsilon=0} \right] d^4x = 0$$

and so we can conclude that the current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi^*}{\partial \epsilon}\bigg|_{\epsilon=0} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \right) \frac{\partial x^*_\nu}{\partial \epsilon}\bigg|_{\epsilon=0}$$

is conserved. In the case of a vector field A^λ this becomes

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \frac{\partial A^{*\lambda}}{\partial \epsilon}\bigg|_{\epsilon=0} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L} \right) \frac{\partial x^*_\nu}{\partial \epsilon}\bigg|_{\epsilon=0}$$

Note that this derivation assumes ε has no indices (i.e. is a scalar). To be more precise we should take into account the possibility that ε may be a vector or even matrix quantity, and write it as ε_α where α stands for any possible index. In this case we should go back to the second from last line of the proof and extract the conserved quantity

$$J^{\mu\alpha}\varepsilon_\alpha = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi^*}{\partial \varepsilon_\alpha}\bigg|_{\varepsilon_\alpha=0} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \right) \frac{\partial x^*_\nu}{\partial \epsilon_\alpha}\bigg|_{\varepsilon_\alpha=0} \right) \varepsilon_\alpha$$

where there is now one conserved current $J^{\mu\alpha}$ for each ε_α (see example iv) below).

4.2 Examples

Let us assume that the field does not change, i.e. $\phi^*(x) = \phi(x)$.

i) Let $\mathcal{L}$ be invariant under time translation, $x^{*0} = x^0 + \varepsilon$, $x^{*i} = x^i$, then

$$J^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} \equiv \mathcal{H}$$

is conserved (this is the Hamiltonian for the field).

ii) Suppose $\mathcal{L}$ invariant under $x^{*\mu} = x^\mu + \varepsilon$, then

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

is conserved (this is the energy-momentum density for the field).

iii) Suppose $\mathcal{L}$ is symmetric under rotations about the x^3 axis, that is,

$$x^{*1} = x^1 \cos\varepsilon + x^2 \sin\varepsilon \qquad x^{*2} = -x^1 \sin\varepsilon + x^2 \cos\varepsilon \qquad x^{*3} = x^3 \qquad x^{*0} = x^0$$

then the angular momentum density of the field is conserved,

$$\frac{\partial \mathcal{L}}{\partial(\partial_3 \phi)} (x^1 \partial_2 \phi - x^2 \partial_1 \phi)$$

iv) Suppose that $\mathcal{L}$ is completely rotationally invariant in the space dimensions. An infinitesimal rotation can be written as

$$x^i \mapsto x^i + \varepsilon_{ij} x^j$$

where $\varepsilon_{ij} = -\varepsilon_{ji}$ is a three by three skew-symmetric real matrix, and so an element of $\mathfrak{so}(3)$ (i.e. a generator of rotations). We then have that

$$-J^{ijk} \varepsilon_{jk} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \partial^l \phi - g^{il} \mathcal{L} \frac{\partial x^*_l}{\partial \epsilon_{jk}} \bigg|_{\varepsilon_{jk}=0} \right) \varepsilon_{jk}$$

is conserved. Now,

$$\frac{\partial x^*_l}{\partial \epsilon_{jk}} = \frac{\partial \, \varepsilon_{ln} x_n}{\partial \epsilon_{jk}} = \delta^j_l \delta^k_n x_n - \delta^j_n \delta^k_l x_n = \delta^j_l x_k - \delta^k_l x_j$$

giving

$$-J^{ijk} \varepsilon_{jk} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \left(\partial^j \phi \, x_k - \partial^k \phi \, x_j \right) - g^{ij} \mathcal{L} x_k + g^{ik} \mathcal{L} x_j \right) \varepsilon_{jk}$$

so we can pick out our conserved currents

$$-J^{ijk} = \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \left(\partial^j \phi \, x_k - \partial^k \phi \, x_j \right) - g^{ij} \mathcal{L} x_k + g^{ik} \mathcal{L} x_j$$

For example, consider rotations about the x_1 axis. Then we have the following conserved currents:

$$-J^{123} = \frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)}\Big(\partial^2\phi\, x_3 - \partial^3\phi\, x_2\Big)$$

$$-J^{112} = \frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)}\Big(\partial^1\phi\, x_2 - \partial^2\phi\, x_1\Big) - \mathcal{L}x_2$$

$$-J^{113} = \frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)}\Big(\partial^1\phi\, x_3 - \partial^3\phi\, x_1\Big) - \mathcal{L}x_3$$

$$J^{132} = -J^{123} \qquad J^{121} = -J^{112} \qquad J^{131} = -J^{113} \qquad J^{111} = J^{112} = J^{133} = 0$$

Note that J^{132} corresponds to the angular momentum density about the x_1 axis.

4.3 Stress-energy tensor

If we assume that our system is invariant under the transformation $x^\nu \to x^\nu + \varepsilon^\nu$ then we have that the tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L}$$

is conserved (note we have changed the λ indices). This tensor is known as the stress-energy tensor.

4.3.1 Stress-energy tensor for electromagnetic field

For a free electromagnetic field, $\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$. The stress energy tensor is given by

$$T^{\mu\nu} = -\frac{1}{16\pi}\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda + \frac{1}{16\pi}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

$$\Rightarrow T^{\mu\nu} = -\frac{1}{4\pi}F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{16\pi}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Although conserved, $T^{\mu\nu}$ is not gauge invariant (as A_λ appears explicitly) or symmetric. We can form a tensor with nicer properties as follows: first, write $F^{\mu\lambda}\partial^\nu A_\lambda = F^{\mu}_{\ \lambda}\partial^\nu A^\lambda = g^{\mu\rho}F_{\rho\lambda}\partial^\nu A^\lambda$ giving

$$T^{\mu\nu} = -\frac{1}{4\pi}g^{\mu\rho}F_{\rho\lambda}\partial^\nu A^\lambda + \frac{1}{16\pi}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

and then substitute in

$$\partial^\nu A^\lambda = F^{\nu\lambda} + \partial^\lambda A^\nu$$

to obtain

$$T^{\mu\nu} = -\frac{1}{4\pi}g^{\mu\rho}F_{\rho\lambda}F^{\nu\lambda} - \frac{1}{4\pi}g^{\mu\rho}F_{\rho\lambda}\partial^\lambda A^\nu + \frac{1}{16\pi}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

and now define

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{4\pi}g^{\mu\rho}F_{\rho\lambda}\partial^\lambda A^\nu$$

$$\Rightarrow \Theta^{\mu\nu} = \frac{1}{4\pi} g^{\mu\rho} F_{\rho\lambda} F^{\lambda\nu} + \frac{1}{16\pi} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

(note interchange of μ and ν to remove the leading minus sign). This tensor is conserved as the difference between it and the stress-energy tensor is $\frac{1}{4\pi} g^{\mu\rho} F_{\rho\lambda} \partial^\lambda A^\nu = \frac{1}{4\pi} F^{\lambda\mu} \partial_\lambda A^\nu$ and

$$\partial_\mu (F^{\lambda\mu} \partial_\lambda A^\nu) = (\partial_\mu F^{\lambda\mu}) \partial_\lambda A^\nu + F^{\lambda\mu} \partial_\mu \partial_\lambda A^\nu = 0$$

where the first term vanishes due to the equations of motion and the second term vanishes as it is the contraction of a symmetric and antisymmetric tensor.

The tensor $\Theta^{\mu\nu}$ is gauge invariant, symmetric, traceless ($\Theta^\mu_\mu = \Theta^{\mu\nu} g_{\mu\nu} = 0$) and can be used to define the angular momentum density $M^{\mu\nu\sigma} = \Theta^{\mu\nu} x^\sigma - \Theta^{\mu\sigma} x^\nu$.

But my very feelings changed to repulsion and terror when I saw the whole man slowly emerge from the window and begin to crawl down the castle wall over the dreadful abyss, face down with his cloak spreading out around him like great wings.

Bram Stoker, *Dracula*

5 Solving Maxwell's equations

We now wish to solve the equation $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$ for a given J^ν. This equation can be written

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \frac{4\pi}{c}J^\nu$$

We impose the Lorenz gauge, $\partial_\mu A^\mu = 0$ so that this becomes

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c}J^\nu$$

This equation will be solved using Green's function methods. Recall that given some differential equation $Df(x) = g(x)$ the Green's function G solves $DG(x) = \delta(x)$; so that $f = \int dx' G(x-x')g(x')$ as then $Df = \int dx' DG(x-x')g(x') = \int dx' \delta(x-x')g(x') = g(x)$.

We will apply Fourier transform methods to obtain the Green's functions we need. The four-dimensional Fourier transform and its inverse are

$$f(x^\mu) = \frac{1}{(2\pi)^2}\int d^4k\, e^{-ik_\nu x^\nu}\tilde{f}(k^\mu) \qquad \tilde{f}(k^\mu) = \frac{1}{(2\pi)^2}\int d^4x\, e^{ik_\nu x^\nu} f(x^\mu)$$

The Fourier representation of the delta-function is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}$$

5.1 Time-independent solutions

First let us obtain solutions with no time dependence. This means we will be solving the equation

$$\vec{\nabla}^2 A^\nu = -\frac{4\pi}{c}J^\nu$$

5.1.1 Green's function

We need to find a Green's function satisfying $\vec{\nabla}^2 G(\vec{x}) = \delta^3(\vec{x})$. In terms of Fourier transforms, this is

$$\vec{\nabla}^2 \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{-i\vec{k}\cdot\vec{x}}\tilde{G}(\vec{k}) = -\frac{1}{(2\pi)^{3/2}}\int d^3k\, \vec{k}^2 e^{-i\vec{k}\cdot\vec{x}}\tilde{G}(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \tilde{G}(\vec{k}) = -\frac{1}{(2\pi)^{3/2}}\frac{1}{\vec{k}^2}$$

giving

$$G(\vec{x}) = -\frac{1}{(2\pi)^3}\int d^3k\, \frac{1}{\vec{k}^2}e^{-i\vec{k}\cdot\vec{x}} = \int_0^\infty d\alpha \int d^3k\, e^{-i\vec{k}\cdot\vec{x}-\alpha\vec{k}^2}$$

as $\int_0^\infty d\alpha\, e^{-\alpha x} = \frac{1}{\alpha}$. Now the k integral can be expressed as a Gaussian:

$$\int d^3k\, e^{-i\vec{k}\cdot\vec{x}-\alpha\vec{k}^2} = \int d^3k\, e^{-\alpha\left(\vec{k}+\frac{i\vec{x}}{2\alpha}\right)^2} e^{-\frac{\vec{x}^2}{4\alpha}} = \left(\frac{\pi}{\alpha}\right)^{3/2} e^{-\frac{\vec{x}^2}{4\alpha}}$$

so we now have

$$G(\vec{x}) = -\frac{1}{(2\pi)^3}\pi^{3/2}\int_0^\infty d\alpha \left(\frac{1}{\alpha}\right)^{3/2} e^{-\frac{\vec{x}^2}{4\alpha}}$$

Let $u = \alpha^{-1/2}$, then $-2du = \alpha^{-3/2}d\alpha$ and we have

$$G(\vec{x}) = \frac{1}{4\pi^{3/2}}\int_\infty^0 du\, e^{-\frac{\vec{x}^2}{4}u^2} = -\frac{1}{4\pi^{3/2}}\frac{1}{2}\sqrt{\frac{4\pi}{\vec{x}^2}}$$

using the fact that the Gaussian integral is symmetric, and hence

$$G(\vec{x}) = -\frac{1}{4\pi|\vec{x}|}$$

5.1.2 Magnetostatic and electrostatic potentials

For an electrostatic system, we have $\vec{\nabla}^2 A^0 = \vec{\nabla}^2\Phi = -4\pi\rho(\vec{x}) = -\frac{4\pi}{c}\sum_e e\,\delta^3(\vec{x}-\vec{x}_e)$, where the sum ranges over the different charges in the system, with $\vec{x}_e$ signifying the position of the charge e. Using the Green's function above we see that

$$\Phi(\vec{x}) = \int d^3x'\,\frac{\rho(\vec{x'})}{|\vec{x}-\vec{x}'|} = \int d^3x'\,\frac{\sum_e e\,\delta^3(\vec{x}'-\vec{x}^e)}{|\vec{x}-\vec{x}'|} = \sum_e \frac{e}{|\vec{x}-\vec{x}_e|}$$

giving

$$\vec{E} = -\vec{\nabla}\Phi = \sum_e e\frac{\vec{x}-\vec{x}_e}{|\vec{x}-\vec{x}_e|^3}$$

Similarly the vector potential is given by

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x'\,\frac{\vec{J}(x')}{|\vec{x}-\vec{x}'|}$$

5.2 Time-dependent solutions

We will now seek solutions of the full equation

$$\partial_\mu\partial^\mu A^\nu = \frac{4\pi}{c}J^\nu$$

5.2.1 Green's function

The Green's function we seek is a solution $D(z^\rho)$ of $\partial_\mu\partial^\mu D(z^\rho) = \delta^4(z^\rho)$ where $z^\rho = x^\rho - x'^\rho$. Appealing to Fourier transforms again, we have

$$\partial_\mu\partial^\mu \frac{1}{(2\pi)^2}\int d^4k\,\tilde{D}(k)e^{-ik_\nu z^\nu} = -\frac{1}{(2\pi)^2}\int d^4k\,k_\mu k^\mu \tilde{D}(k^\mu)e^{-ik_\nu z^\nu} = \frac{1}{(2\pi)^4}\int d^4k\,e^{-ik_\nu z^\nu}$$

from which

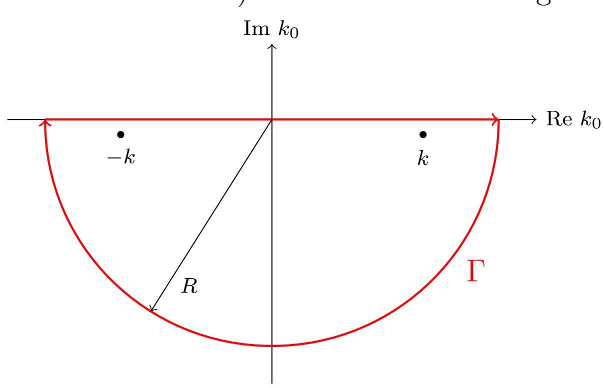
$$\tilde{D}(k^\mu) = -\frac{1}{4\pi^2}\frac{1}{k_\mu k^\mu} = -\frac{1}{4\pi}\frac{1}{k_0^2 - k^2}$$

where $k = |\vec{k}|$. We now must solve

$$D(z^\mu) = -\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k_0^2-k^2}e^{-ik_\nu z^\nu} = -\frac{1}{(2\pi)^4}\int d^3k\,e^{i\vec{k}\cdot\vec{z}}\int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2-k^2}e^{-ik_0 z^0}$$

Note the sign change in the first exponential when converting to three-dimensions. Now, to evaluate the k_0 integral we use contour integration, treating k_0 as a complex number, $k_0 = \mathrm{Re}\,k_0 + i\mathrm{Im}\,k_0$. This gives $e^{i\vec{k}\cdot\vec{z}} = e^{\mathrm{Im}\,k_0 z^0}e^{-i\mathrm{Re}\,k_0}$. So that the integral converges we must choose $\mathrm{Im}\,k_0 < 0$ for $z^0 > 0$. This condition is imposed as $z^0 = x^0 - x'^0 = c(t-t')$, and so positive z^0 ensures that contributions to the Green's function and hence to the potential A^μ only come from events that occur at times $t' < t$, i.e. events in the past. Thus causality is ensured.

Now, the poles of the integrand are $\pm k$ and so lie on the real axis. To avoid them, we displace our contour by an infinitesimal amount $i\epsilon$ so that it lies just in the upper-half plane (formally we should let $\epsilon \to 0$ at the end). Our contour of integration Γ then looks like:



We than have

$$\oint_\Gamma dk_0 \frac{e^{-ik_0 z^0}}{k_0^2-k^2} = \int_{-R}^{R} dk_0 \frac{e^{-ik_0 z^0}}{k_0^2-k^2} + \int_{\text{semi-circle}} \frac{e^{-ik_0 z^0}}{k_0^2-k^2}$$

Now on the semicircle we can write $k_0 = Re^{-i\varphi} = R\cos\varphi - iR\sin\varphi$, so that the integral becomes an integral over φ from 0 to π. So we have

$$\int_{\text{semi-circle}} \frac{e^{-ik_0 z^0}}{k_0^2-k^2} = -i\int_0^\pi d\varphi\,Re^{i\varphi}e^{-Rz^0\sin\varphi - iRz^0\cos\varphi}\frac{1}{R^2e^{-2i\varphi}-k^2}$$

and $\sin\varphi \geq 0$ for this range of φ, hence the integrand goes to zero as $R \to \infty$, as required. The value of the k_0 integral we are interested in will hence be given by the $-2\pi i$ times the sum of the residues inside the contour. The residues are:

$$\lim_{k_0 \to k} (k_0 - k) \frac{e^{-ik_0 z^0}}{(k_0 - k)(k_0 + k)} = \frac{e^{-ikz^0}}{2k}$$

$$\lim_{k_0 \to -k} (k_0 + k) \frac{e^{-ik_0 z^0}}{(k_0 - k)(k_0 + k)} = -\frac{e^{ikz^0}}{2k}$$

hence

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2 - k^2} e^{-ik_0 z^0} = \frac{\pi i}{k} \left(e^{ikz^0} - e^{-ikz^0} \right) \Theta(z^0)$$

where the Heaviside function

$$\Theta(z^0) = \begin{cases} 1 & z^0 > 0 \\ 0 & z^0 < 0 \end{cases}$$

is added as the residues only contribute for positive z^0. So we now have

$$\begin{aligned} D(z^\mu) &= \frac{\Theta(z^0)}{(2\pi)^4} \pi i \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \frac{1}{k} \left(e^{ikz^0} - e^{-ikz^0} \right) \\ &= \frac{\Theta(z^0)}{16\pi^3} i \int_0^\infty dk \, k^2 \frac{1}{k} \left(e^{ikz^0} - e^{-ikz^0} \right) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \, e^{ikz\cos\theta} \end{aligned}$$

upon switching to polar coordinates and choosing the coordinate frame such that the k^3 axis makes an angle of θ with $\vec{z}$, and letting $z = |\vec{z}|$. Integrating over the angles, we get

$$\begin{aligned} D(z^\mu) &= \frac{\Theta(z^0)}{8\pi^2} i \int_0^\infty dk \, k \left(e^{ikz^0} - e^{-ikz^0} \right) \left(\frac{e^{ikz}}{ikz} - \frac{e^{-ikz}}{ikz} \right) \\ &= -\frac{\Theta(z^0)}{8\pi^2 z} \int_0^\infty dk \left(e^{ik(z^0+z)} + e^{-ik(z^0+z)} - e^{ik(z^0-z)} - e^{-ik(z^0-z)} \right) \end{aligned}$$

but if we let $k \mapsto -k$

$$\int_0^\infty dk \, e^{-ik(z^0+z)} = \int_0^{-\infty} d(-k) e^{ik(z^0+z)} = \int_{-\infty}^0 dk \, e^{ik(z^0+z)}$$

hence

$$D(z^\mu) = -\frac{\Theta(z^0)}{8\pi^2 z} \int_{-\infty}^\infty dk \left(e^{ik(z+z^0)} - e^{ik(z^0-z)} \right) = \frac{\Theta(z^0)}{4\pi z} \left(\delta(z^0 - z) - \delta(z^0 + z) \right)$$

remembering the integral representation of the delta function. Owing to the Heaviside function only the first delta function will contribute, so

$$D_{ret}(z^\mu) = \frac{\Theta(z^0)}{4\pi z} \delta(z^0 - z)$$

The subscript signifies that this is the *retarded* Green's function (that is, the Green's function resulting from the contribution of events in the past).

We can put the Green's function in covariant form using

$$\delta(z_\mu z^\mu) = \delta([z_0 - z][z_0 + z]) = \frac{\delta(z_0 - z)}{|z_0 + z|} + \frac{\delta(z_0 + z)}{|z_0 - z|}$$

as $\delta(ab) = \frac{\delta(a)}{|b|} + \frac{\delta(b)}{|a|}$. Hence

$$\Theta(z^0)\delta(z_\mu z^\mu) = \Theta(z^0)\frac{\delta(z_0 - z)}{|z_0 + z|} = \Theta(z^0)\frac{\delta(z_0 - z)}{|2z|}$$

and so

$$D_{ret}(z^\mu) = \frac{\Theta(z^0)}{2\pi}\delta(z_\mu z^\mu)$$

Recalling that $z^\mu = x^\mu - x'^\mu$ we can state our final results for the Green's function as

$$D_{ret}(x^\mu - x'^\mu) = \frac{\Theta(x^0 - x'^0)}{4\pi z}\delta(x^0 - x'^0 - |\vec{x} - \vec{x'}|)$$

and in covariant form,

$$D_{ret}(x^\mu - x'^\mu) = \frac{\Theta(x^0 - x'^0)}{2\pi}\delta([x_\mu - x'_\mu][x^\mu - x'^\mu])$$

Note that if we had taken $z^0 < 0$ and closed our contour in the upper half-plane (with poles displaced upwards) we would have obtained the *advanced* Green's function

$$D_{adv}(x^\mu - x'^\mu) = \frac{\Theta(x'^0 - x^0)}{4\pi z}\delta(x^0 - x'^0 + |\vec{x} - \vec{x'}|)$$

5.2.2 Lienard-Wiechart potentials

Of night and light and the half-light.

W.B. Yeats, "He Wishes For The Cloths Of Heaven"

We can now work out the potentials A^μ that solve the Maxwell equation $\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$. They are given by

$$A^\mu(x^\sigma) = \frac{4\pi}{c}\int d^4x' D_{ret}(x - x')J^\mu(x'^\sigma)$$

where

$$J^\mu(x'^\sigma) = e\frac{dx'^\sigma}{dt'}\delta^3(\vec{x}' - \vec{x}'_e(t))$$

or in covariant form

$$J^\mu(x'^\sigma) = ec\int d\tau \frac{dx'^\sigma}{d\tau}\delta^4(x'^\sigma - x'^\sigma_e(\tau))$$

as integrating over τ and using that

$$\delta(f(\tau)) = \sum_{i:f(\tau_i)=0} \frac{\delta(\tau - \tau_i)}{|f'(\tau_i)|}$$

we have

$$J^\mu(x'^\sigma) = ec\frac{dx'^\sigma}{d\tau}\Big|_{\tau'} \delta^3(\vec{x}' - \vec{x}'_e(t)) \left(\frac{dx'^0}{d\tau}\right)^{-1}\Big|_{\tau'}$$

where $x'^0 - x^0_e(\tau') = 0$. Now, $\frac{dx'^0}{d\tau}|_{\tau'} = \frac{dx'^0}{dt}\frac{dt}{d\tau}|_{\tau'} = c\frac{dt}{d\tau}|_{\tau'}$ and $\frac{dx'^\sigma}{d\tau}|_{\tau'} = \frac{dx'^\sigma}{dt}\frac{dt}{d\tau}|_{\tau'}$ so this reduces to the local form.

So substituting this in, we have

$$\begin{aligned} A^\mu(x^\sigma) &= \frac{4\pi}{c}\int d\tau d^4x' \frac{\Theta(x^0 - x'^0)}{2\pi}\delta([x_\sigma - x'_\sigma][x^\sigma - x'^\sigma])ec\frac{dx'^\sigma}{d\tau}\delta^4(x'^\sigma - x'^\sigma_e(\tau)) \\ &= 2e\int d\tau \Theta(x^0 - x^0_e)\delta([x_\sigma - x^e_\sigma(\tau)][x^\sigma - x^\sigma_e(\tau)])V^\mu(\tau) \end{aligned}$$

where we have written $V^\mu \equiv \frac{dx^\mu_e}{d\tau}$. Now, we need the roots of the argument of the delta function:

$$[x_\sigma - x^e_\sigma(\tau)][x^\sigma - x^\sigma_e(\tau)] = 0 \Rightarrow (x_0 - x^e_0(\tau))^2 - |\vec{x} - \vec{x}_e(\tau)|^2 = 0$$

There are two possibilities:

$$x_0 - x^e_0(\tau) = \pm|\vec{x} - \vec{x}_e(\tau)|$$

The Heaviside function constrains us to choose the positive option. We see that the unique root of $[x_\sigma - x^e_\sigma(\tau)][x^\sigma - x^\sigma_e(\tau)] = 0$ which ensures causality is given by the **retarded time** τ_0:

$$x_0 - x^e_0(\tau_0) = |\vec{x} - \vec{x}_e(\tau_0)|$$

Physically, the retarded time gives the unique time at which the charged particle intersects the past light-cone of the observation point. Now, as

$$\frac{d}{d\tau}[x_\sigma - x^e_\sigma(\tau)][x^\sigma - x^\sigma_e(\tau)] = -2[x_\sigma - x^e_\sigma(\tau)]\frac{d}{d\tau}x^\sigma_e(\tau)$$

We then have that

$$\Theta(x^0 - x^0_e)\delta([x_\sigma - x^e_\sigma(\tau)][x^\sigma - x^\sigma_e(\tau)]) = \frac{\delta(\tau - \tau_0)}{2[x_\sigma - x^e_\sigma(\tau_0)]V^\sigma(\tau_0)}$$

so, writing $R^\sigma = x^\sigma - x^\sigma_e(\tau_0)$, we have

$$A^\mu(x^\sigma) = \frac{V^\mu(\tau)}{R_\sigma(\tau)V^\sigma(\tau)}\Bigg|_{\tau_0}$$

Note that the retarded time is in this notation defined by $R_\sigma(\tau_0)R^\sigma(\tau_0) = 0$ and that then $R^0 = R = |\vec{R}| = |\vec{x} - \vec{x}^\mu_e(\tau_0)|$.

5.2.3 Electromagnetic fields from Lienard-Wiechart potentials: method one

We wish to evaluate the electromagnetic fields $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ arising from the motion of a charged particle. Consider the integral expression for the potential

$$A^\nu(x^\sigma) = 2e \int d\tau \Theta(x^0 - x_e^0)\delta([x_\sigma - x_\sigma^e(\tau)]^2)V^\nu(\tau)$$

We will differentiate this with respect to x^μ. First, note that

$$\partial_\mu \Theta(x^0 - x_e^0) = \delta(x^0 - x_e^0)$$

and this will give a term $\delta(-|\vec{x} - \vec{x}^e(\tau)]^2)$ which only contributes for $\vec{x} = \vec{x}^e(\tau)$ and can be neglected. Thus we have

$$\partial^\mu A^\nu = 2e \int d\tau \Theta(x^0 - x_e^0)\partial^\mu \delta([x_\sigma - x_\sigma^e(\tau)]^2)V^\nu(\tau)$$

Let us now write

$$\begin{aligned}
\partial^\mu \delta([x_\sigma - x_\sigma^e(\tau)]^2) &= \partial^\mu \delta(R_\sigma(\tau)R^\sigma(\tau)) \\
&= \frac{\partial}{\partial R_\sigma R^\sigma}\delta(R_\sigma(\tau)R^\sigma(\tau))\partial^\mu(R_\sigma R^\sigma) \\
&= \frac{d}{d\tau}\delta(R_\sigma(\tau)R^\sigma(\tau))\frac{\partial\tau}{\partial R_\sigma R^\sigma}\partial^\mu(R_\sigma R^\sigma)
\end{aligned}$$

and as $R^\sigma = x^\sigma - x_e^\sigma(\tau)$ we have

$$\partial^\mu(R_\sigma R^\sigma) = 2R^\mu$$

and

$$\frac{d}{d\tau}(R_\sigma R^\sigma) = -2R_\sigma V^\sigma \Rightarrow \frac{d\tau}{dR_\sigma R^\sigma} = -\frac{1}{2R_\sigma V^\sigma}$$

so (using ρ as our dummy variable in the denominator as σ is used in the numerator)

$$\partial^\mu \delta([x_\sigma - x_\sigma^e(\tau)]^2) = -\frac{R^\mu}{R_\rho V^\rho}\frac{d}{d\tau}\delta(R_\sigma(\tau)R^\sigma(\tau))$$

Thus

$$\partial^\mu A^\nu = -2e \int d\tau \Theta(x^0 - x_e^0)\frac{R^\mu V^\nu}{R_\rho V^\rho}\frac{d}{d\tau}\delta(R_\sigma(\tau)R^\sigma(\tau))$$

and we can integrate this by parts to obtain

$$\partial^\mu A^\nu = 2e \int d\tau \Theta(x^0 - x_e^0)\delta(R_\sigma(\tau)R^\sigma(\tau))\frac{d}{d\tau}\left(\frac{R^\mu V^\nu}{R_\rho V^\rho}\right)$$

Recalling from the derivation of the potentials that

$$\Theta(x^0 - x_e^0)\delta(R_\sigma(\tau)R^\sigma(\tau)) = \frac{\delta(\tau - \tau_0)}{2R_\sigma(\tau_0)V^\sigma(\tau_0)}$$

we find

$$\partial^\mu A^\nu = \frac{e}{R_\sigma V^\sigma}\frac{d}{d\tau}\left(\frac{R^\mu V^\nu}{R_\rho V^\rho}\right)$$

with this evaluated at the retarded time. Carrying out the differentiation, with a dot denoting a derivative with respect to proper time,

$$\partial^\mu A^\nu = \frac{e}{R_\sigma V^\sigma}\left(\frac{-V^\mu V^\nu + R^\mu \dot{V}^\nu}{R_\rho V^\rho} - \frac{R^\mu V^\nu}{(R_\rho V^\rho)^2}\left(-V_\lambda V^\lambda + R_\lambda \dot{V}^\lambda\right)\right)$$

and hence

$$F^{\mu\nu} = \frac{eV_\rho V^\rho}{(R_\sigma V^\sigma)^3}\left(R^\mu V^\nu - R^\nu V^\mu\right) + \frac{e}{(R_\sigma V^\sigma)^2}\left(R^\mu \dot{V}^\nu - R^\nu \dot{V}^\mu\right) - \frac{eR_\rho \dot{V}^\rho}{(R_\sigma V^\sigma)^3}\left(R^\mu V^\nu - R^\nu V^\mu\right)$$

Note that we can write $F^{\mu\nu}$ as a sum of two parts, one a velocity field containing terms depending on V^μ and the other an acceleration or radiative field containing derivatives of the velocity, $\dot{V}^\mu$:

$$F^{\mu\nu} = F^{\mu\nu}_{vel} + F^{\mu\nu}_{rad}$$

where

$$F^{\mu\nu}_{vel} = \frac{eV_\rho V^\rho}{(R_\sigma V^\sigma)^3}\left(R^\mu V^\nu - R^\nu V^\mu\right)$$

and

$$F^{\mu\nu}_{rad} = \frac{e}{(R_\sigma V^\sigma)^2}\left(R^\mu \dot{V}^\nu - R^\nu \dot{V}^\mu\right) - \frac{eR_\rho \dot{V}^\rho}{(R_\sigma V^\sigma)^3}\left(R^\mu V^\nu - R^\nu V^\mu\right)$$

5.2.4 Electromagnetic fields from Lienard-Wiechart potentials: method two

Consider the covariant form of the equation defining the retarded time:

$$R_\sigma(\tau_0) R^\sigma(\tau_0) = 0$$

This equation defines τ_0 as a function of the observation point x^μ. To find how τ_0 changes with x^μ we differentiate

$$\partial_\mu(R_\sigma(\tau_0)R^\sigma(\tau_0)) = 0 \Rightarrow 2R_\sigma \partial_\mu R^\sigma = 0$$

Now,

$$\partial_\mu R^\sigma = \partial_\mu x^\sigma - \partial_\mu x^e(\tau) = \delta^\sigma_\mu - \frac{d}{d\tau}x^\sigma_e(\tau)\partial^\mu \tau = \delta^\sigma_\mu - V^\mu \partial_\mu \tau$$

so at the retarded time

$$R_\sigma(\delta^\sigma_\mu - V^\mu \partial_\mu \tau_0) = 0$$

and hence

$$\partial_\mu \tau_0 = \frac{R_\mu}{R_\sigma V^\sigma} \Rightarrow \partial^\mu \tau_0 = \frac{R^\mu}{R_\sigma V^\sigma}$$

with the expression on the right evaluated at the retarded time.

We will use this to now evaluate the electromagnetic field tensor resulting from the four-potential

$$A^\nu(x^\rho) = \left.\frac{eV^\nu}{V^\sigma R_\sigma}\right|_{\tau_0}$$

In what follows we will denote differentiation with respect to proper time by a dot. Note that our expressions must all be evaluated at the retarded time - so in effect we will work out $\partial^\mu A^\nu$ treating R_σ and V_σ as functions of τ with the awareness that in reality everything we do is evaluated at τ_0. This allows us to substitute the above expression for $\partial^\mu \tau_0$ for $\partial^\mu \tau$ as we derive. So we have

$$\begin{aligned}
\partial^\mu A^\nu &= e\frac{\partial^\mu V^\nu}{V_\sigma R_\sigma} - \frac{eV^\nu}{(V^\sigma R_\sigma)^2}\Big((\partial^\mu V^\rho)R_\rho + V^\rho(\partial^\mu R_\rho)\Big) \\
&= e\frac{\dot{V}^\nu \partial^\mu \tau}{V_\sigma R_\sigma} - \frac{eV^\nu}{(V^\sigma R_\sigma)^2}\Big(\dot{V}^\rho R_\rho \partial^\mu \tau + V^\rho(\delta^\mu_\rho - V_\rho \partial^\mu \tau)\Big) \\
&= e\frac{\dot{V}^\nu R^\mu}{(V_\sigma R_\sigma)^2} - \frac{eV^\nu}{(V^\sigma R_\sigma)^2}\left(\frac{\dot{V}^\rho R_\rho R^\mu}{V^\sigma R_\sigma} + V^\mu - \frac{V^\rho V_\rho R^\mu}{R_\sigma V^\sigma}\right) \\
&= e\frac{R^\mu V^\nu V^\rho V_\rho}{(V^\sigma R_\sigma)^3} - \frac{eV^\mu V^\nu}{(V^\sigma R_\sigma)^2} + \frac{eR^\mu \dot{V}^\nu}{(V^\sigma R_\sigma)^2} - \frac{eR^\mu V^\nu \dot{V}^\rho R_\rho}{(V^\sigma R_\sigma)^3}
\end{aligned}$$

Hence we again find we can write $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as a sum of two parts, one involving terms containing the four-velocity of the charge V^μ and the other involving terms involving the acceleration $\dot{V}^\mu$, that is,

$$F^{\mu\nu} = F^{\mu\nu}_{vel} + F^{\mu\nu}_{rad}$$

where

$$F^{\mu\nu}_{vel} = \frac{ec^2}{(V^\sigma R_\sigma)^3}\Big(R^\mu V^\nu - R^\nu V^\mu\Big)$$

and

$$F^{\mu\nu}_{rad} = \frac{e}{(V^\sigma R_\sigma)^2}\Big(R^\mu \dot{V}^\nu - R^\nu \dot{V}^\mu\Big) - \frac{e\dot{V}^\rho R_\rho}{(V^\sigma R_\sigma)^3}\Big(R^\mu V^\nu - R^\nu V^\mu\Big)$$

where we have used that $V^\rho V_\rho = \gamma^2 c^2 - \gamma^2 v^2 = \gamma^2 c^2(1 - v^2/c^2) = c^2$.

5.2.5 Properties of the electromagnetic fields due to a moving charge

We can immediately work out some important properties of the velocity and radiative fields. Consider

$$R_\mu F^{\mu\nu}_{vel} = -\frac{ec^2}{(V^\sigma R_\sigma)^3} R^\nu V^\mu \neq 0$$

where we have used that $R_\mu R^\mu = 0$ at the retarded time (remember that all our expressions for the fields are evaluated at τ_0), and

$$R_\mu F^{\mu\nu}_{rad} = -\frac{eR_\mu R^\nu \dot{V}^\mu}{(V^\sigma R_\sigma)^2} + \frac{e\dot{V}^\rho R_\rho R_\mu R^\nu V^\mu}{(V^\sigma R_\sigma)^3} = 0$$

upon making a single fraction and switching some of the dummy variables to agree. Setting ν in $R_\mu F^{\mu\nu}_{acc}$ to zero and j successively we find

$$R_\mu F^{\mu 0}_{rad} = 0 \Rightarrow R_i F^{i0}_{rad} = 0 \Rightarrow R^i E^i_{rad} = 0 \Rightarrow \vec{n}\cdot\vec{E}_{rad} = 0$$

$$R_\mu F^{\mu j}_{rad} = R_0 F^{0j}_{rad} + R_i F^{ij}_{rad} = -RE^j + R^i\varepsilon^{ijk}B^k = 0 \Rightarrow E^j = -\varepsilon^{jik}\frac{R^i}{R}B^k \Rightarrow \vec{E} = -\vec{n}\times\vec{B}$$

where $\vec{n}$ is a unit direction in the direction of $\vec{R}$. Note that this implies $\vec{n}$, $\vec{E}_{rad}$ and $\vec{B}_{rad}$ are mutually orthogonal and have the same magnitude (in Gaussian/Heaviside-Lorentz units).

We can similarly consider the dual tensor, $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$. For both $F^{\mu\nu}_{vel}$ and $F^{\mu\nu}_{rad}$ we see every term in $R_\alpha\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}R_\alpha F_{\mu\nu}$ will contain $\epsilon^{\alpha\beta\mu\nu}R_\alpha R_\mu$ or $\epsilon^{\alpha\beta\mu\nu}R_\alpha R_\nu$ and hence contract to zero. Thus,

$$R_\mu\tilde{F}^{\mu\nu} = 0$$

and hence

$$\vec{n}\cdot\vec{B} = 0 \qquad \vec{B} = \vec{n}\times\vec{E}$$

5.2.6 Local form of the electromagnetic fields due to a moving charge

To transform our covariant expressions for $F^{\mu\nu}_{vel}$ and $F^{\mu\nu}_{rad}$ to our local frame we recall that $V^\mu = \gamma(c,\vec{v}) = \gamma(c, c\vec{\beta})$, where $\vec{v} = c\vec{B}$. We need to work out $\dot{V}^\mu = \frac{d}{d\tau}V^\mu$. Now, $\frac{d}{d\tau} = \gamma\frac{d}{dt}$, so we have

$$\frac{d}{d\tau}\gamma c = c\gamma\frac{d}{dt}\frac{1}{\sqrt{1-\vec{\beta}^2}} = c\gamma\frac{\vec{\beta}\cdot\vec{\alpha}}{\sqrt{1-\vec{\beta}^2}^3} = c\gamma^4\vec{\beta}\cdot\vec{\alpha}$$

where $\vec{\alpha} = \frac{d}{dt}\vec{\beta}$, and

$$\frac{d}{d\tau}\gamma c\beta = c\gamma\frac{d}{dt}\gamma\beta = c\gamma\left(\vec{\beta}\frac{d}{dt}\gamma + \gamma\frac{d}{dt}\vec{\beta}\right) = c\gamma\left(\vec{\beta}\gamma^3(\vec{\beta}\cdot\vec{\alpha}) + \gamma\vec{\alpha}\right) = c\gamma^4(\vec{\beta}\cdot\vec{\alpha})\vec{\beta} + c\gamma^2\vec{\alpha}$$

We also have that $V^\sigma R_\sigma = R\gamma c - \gamma\vec{V}\cdot\vec{R} = \gamma cR(1-\vec{n}\cdot\vec{\beta})$. Hence we have

$$\begin{aligned} E^i_{vel} = F^{i0}_{vel} &= \frac{ec^2}{(V^\sigma R_\sigma)^3}\left(R^iV^0 - R^0V^i\right) \\ &= \frac{ec^2}{\gamma^3c^3R^3(1-\vec{n}\cdot\vec{\beta})^3}\left(\gamma cR^i - R\gamma c\beta^i\right) \end{aligned}$$

so

$$\vec{E}_{vel} = \frac{e(\vec{n}-\vec{\beta})}{\gamma^2 R^2 (1-\vec{n}\cdot\vec{\beta})^3}\bigg|_{t_0}$$

$$\vec{B}_{vel} = \vec{n}\times\vec{E}_{vel}$$

Turning to the radiative fields, we have

$$E^i_{rad} = F^{i0}_{rad} = \frac{e}{(V^\sigma R_\sigma)^2}\left(R^i\dot{V}^0 - R^0\dot{V}^i\right) - \frac{e\dot{V}^\rho R_\rho}{(V^\sigma R_\sigma)^3}\left(R^i V^0 - R^0 V^i\right)$$

so, writing $\vec{R} = R\vec{n}$,

$$\begin{aligned}
\frac{\gamma^3 c^3 R^3 (1-\vec{n}\cdot\vec{\beta})^3}{e}\vec{E}_{rad} &= \gamma c R\left(1-\vec{n}\cdot\vec{\beta}\right)\left(\gamma^4 cR\vec{n}\vec{\beta}\cdot\vec{\alpha} - R\left[c\gamma^4(\vec{\beta}\cdot\vec{\alpha})\vec{\beta} + c\gamma^2\vec{\alpha}\right]\right) \\
&\quad - \left(\gamma^4 cR\vec{\beta}\cdot\vec{\alpha} - \gamma^4 cR(\vec{\beta}\cdot\vec{\alpha})\vec{n}\cdot\vec{\beta} - \gamma^2 cR\vec{n}\cdot\vec{\alpha}\right)\left(c\gamma R\vec{n} - c\gamma R\vec{\beta}\right) \\
&= \gamma^5 c^2 R^2\left[\left(1-\vec{n}\cdot\vec{\beta}\right)\left(\vec{\beta}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - \frac{\vec{\alpha}}{\gamma^2}\right)\right. \\
&\quad \left. - \left(\vec{\beta}\cdot\vec{\alpha}(1-\vec{n}\cdot\vec{\beta}) - \frac{\vec{n}\cdot\vec{\alpha}}{\gamma^2}\right)\left(\vec{n}-\vec{\beta}\right)\right] \\
&= \gamma^5 c^2 R^2\left[\frac{1}{\gamma^2}\vec{\alpha}(1-\vec{n}\cdot\vec{\beta}) + (\vec{n}-\vec{\beta})\Big(\vec{\beta}\cdot\vec{\alpha}(1-\vec{n}\cdot\vec{\beta})\right. \\
&\quad \left. - \vec{\beta}\cdot\vec{\alpha}(1-\vec{n}\cdot\vec{\beta}) + \frac{1}{\gamma^2}\vec{n}\cdot\vec{\alpha}\right] \\
&= \gamma^3 c^2 R^2\left[(\vec{n}-\vec{\beta})\vec{n}\cdot\vec{\alpha} - \vec{\alpha}(1-\vec{n}\cdot\vec{\beta})\right]
\end{aligned}$$

hence

$$\vec{E}_{rad} = \frac{e}{cR}\frac{\vec{n}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - \vec{\alpha}(1-\vec{n}\cdot\vec{\beta})}{(1-\vec{n}\cdot\vec{\beta})^3}\bigg|_{t_0}$$

$$\vec{B}_{rad} = \vec{n}\times\vec{E}_{rad}$$

We can use the vector identity $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ with $\vec{a}=\vec{n}$, $\vec{b}=\vec{n}-\vec{\beta}$ and $\vec{c}=\vec{\alpha}$ to see that

$$\vec{n}\times([\vec{n}-\vec{\beta}]\times\vec{\alpha}) = \vec{n}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - \vec{\alpha}(1-\vec{n}\cdot\vec{\beta})$$

so we can write the radiative electric field as

$$\vec{E}_{rad} = \frac{e}{Rc}\frac{\vec{n}\times([\vec{n}-\vec{\beta}]\times\vec{\alpha})}{(1-\vec{n}\cdot\vec{\beta})^3}\bigg|_{t_0}$$

6 Power radiated by accelerating charge

The essential idea is that the energy flux per unit area in the direction $\vec{n}$ is given by $\vec{S}\cdot\vec{n}$ where $\vec{S} = \frac{c}{4\pi}\vec{E}_{rad}\times\vec{B}_{rad}$ is the Poynting vector. We have that $\vec{E}_{rad}$ is perpendicular to $\vec{B}_{rad}$ and $\vec{n}$ is perpendicular to both with $\vec{B}_{rad} = \vec{n}\times\vec{E}_{rad}$, so

$$|\vec{n}\cdot\vec{S}| = \frac{c}{4\pi}|\vec{E}_{rad}|^2$$

Now the differential power radiated into a solid angle element $d\Omega$ in the direction $\vec{n}$ is

$$dP = R^2|\vec{n}\cdot\vec{S}|d\Omega$$

This expression is in terms of the time t at the observation point; it is often more convenient to work with the time t' in the charge's own frames. We can write the energy radiated between times t_1 and t_2 as

$$\mathcal{E} = \int_{t_1}^{t_2} |\vec{n}\cdot\vec{S}|dt\, d\Omega R^2 = \int_{t_1'}^{t_2'} |\vec{n}\cdot\vec{S}|\frac{dt}{dt'}dt' d\Omega R^2$$

so we see that we should define

$$dP(t') = R^2|\vec{n}\cdot\vec{S}|d\Omega\frac{dt}{dt'} = R^2\frac{c}{4\pi}|\vec{E}_{rad}|^2 d\Omega\frac{dt}{dt'}$$

To evaluate the derivative, we use that

$$ct - ct' = R$$

from the definition of the retarded time. As $R = |\vec{R}| = |\vec{x} - \vec{x}_e(t')$ we find

$$\frac{dt}{dt'} - 1 = -\frac{\vec{R}\cdot\vec{v}}{Rc} \Rightarrow \frac{dt}{dt'} = 1 - \vec{n}\cdot\vec{\beta}$$

and hence

$$\frac{dP(t')}{d\Omega} = R^2\frac{c}{4\pi}|\vec{E}_{rad}|^2(1-\vec{n}\cdot\vec{\beta})$$

and using the expression for $\vec{E}_{rad}$ evaluated in the preceding section we have

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\vec{n}\times([\vec{n}-\vec{\beta}]\times\vec{\alpha})\right|^2}{(1-\vec{n}\cdot\vec{\beta})^5}$$

with this expression evaluated at the retarded time.

An important consequence of the electromagnetic radiation of an accelerating charged particle is that classically an electron turning in circular motion will lose energy due to radiation, leading to a decay of its orbit.

The remainder of this section of the course is not covered by these notes.

Because I do not hope to turn again
Because I do not hope
Because I do not hope to turn

T.S. Eliot, "Ash Wednesday"

7 Bibliography

- Obviously most of the material above was taken from my notes from Dr Buttimore's lectures and shamelessly LaTeXed out under my own name. However I can claim full credit for any mistakes that have appeared, and would appreciate any corrections/suggestions to cblair[at]maths.tcd.ie.

- The covariant derivation of the Lienard-Wiechart potentials was borrowed from *Electrodynamics* by Fulvio Melia and *Classical Electrodynamics* by Jackson (note that the first method for obtaining the electromagnetic fields from the potentials is taken from Jackson while the second was contributed to the 432 course by former students).

- *The Classical Theory of Fields* by Landau and Lifshitz.

- There are some good exam-oriented notes by Eoin Curran at `http://peelmeagrape.net/eoin/notes/fields.pdf`.

- The quotations throughout these notes were all mentioned in lectures by Dr Buttimore; their precise relevance is left as an exercise for the reader. The text of *Dracula* by Bram Stoker is available online at `http://classiclit.about.com/library/bl-etexts/bstoker/bl-bsto-drac-1.htm`. The poems by Yeats and Eliot can presumably also be found online, or in any major collection of their work.