Emmy Noether on Energy Conservation in General Relativity

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Abstract: During the First World War, the status of energy conservation in general relativity was one of the most hotly debated questions surrounding Einstein's new theory of gravitation. His approach to this aspect of general relativity differed sharply from another set forth by Hilbert, even though the latter conjectured in 1916 that both theories were probably equivalent. Rather than pursue this question himself, Hilbert chose to charge Emmy Noether with the task of probing the mathematical foundations of these two theories. Indirect references to her results came out two years later when Klein began to examine this question again with Noether's assistance. Over several months, Klein and Einstein pursued these matters in a lengthy correspondence, which culminated with several publications, including Noether's now famous paper "Invariante Variationsprobleme" [Noether 1918b]. The present account focuses on the earlier discussions from 1916 involving Einstein, Hilbert, and Noether. In these years, a Swiss student named R.J. Humm was studying relativity in Göttingen, during which time he transcribed part of Noether's lost manuscript on Hilbert's invariant energy vector. By making use of this 9-page manuscript, it is possible to reconstruct the arguments Noether set forth in response to Hilbert's conjecture. Her results turn out to be closely related to the findings Klein published two years later, thereby highlighting, once again, how her work significantly deepened contemporary understanding of the mathematical underpinnings of general relativity.

1. Introduction

Emmy Noether's paper "Invariante Variationsprobleme" [Noether 1918b] is regarded today as one of her most important works, especially in view of its relevance for mathematical physics [Uhlenbeck 1983]. Those familiar with her many other achievements might wonder why these have largely been cast in the shadows by Noether's Theorem, the famous result accounting for the relationship between symmetries in physical systems and their related conservation laws.[1] To be sure, standard accounts of Emmy Noether's life have never claimed that her 1918 paper was particularly significant, and for good reason. Her influence and eventual fame as a mathematician had virtually nothing to do with physics or the calculus of variations; these stemmed instead from her contributions to modern algebra.[2] Considering these circumstances, it is natural to ask what motivated Noether to take up this topic in the first place. [Rowe 1999] deals with how Noether's paper arose from discussions in Göttingen concerning the status of energy conservation laws in general relativity. This paper focusses on an earlier discussion that arose in 1916 after Albert Einstein and David Hilbert published their first papers addressing the role of energy conservation in general relativity.

As will be shown below, the approaches taken by Einstein and Hilbert to this aspect of the theory differed strikingly. Hilbert's short paper [Hilbert 1915] was written in great haste and afterward substantially revised when he read the page proofs. Its contents baffled many readers, including Einstein. In 1918, Emmy Noether was working closely with Felix Klein, who was determined to decipher the mathematical meaning of Hilbert's invariant energy vector. Allusions to Noether's role in earlier discussions with Hilbert can be found in Klein's published papers, [Klein 1918a] and [Klein 1918b]. Little has been written, however, about what Noether contributed to these conversations from 1916, mainly due to lack of documentary evidence that might shed more light on her activities during the war years. It is my hope that the present paper will help to clarify an important episode in this story. Here I will mainly focus on her efforts, starting in early 1916, to assist Hilbert's researches on general relativity, while touching only briefly on her subsequent work with Klein, which culminated with the publication of [Noether 1918b].

In the course of exploring the foundations of his general theory of relativity, Einstein had experimented with variational methods [Einstein 1914]. Hilbert was the first, however, to use a variational principle to derive fully covariant gravitational field equations in the form of Euler-Lagrange equations. He published this result in the first of his two papers on "The Foundations of Physics" [Hilbert 1915]. There he emphasized that the resulting system of 14 field equations was *not* independent; instead it satisfied four identical relations, which he interpreted as establishing a linkage between gravity and electromagnetism. However, the precise nature of these relations, and in particular their physical significance, remained obscure up until the publication of [Noether 1918b], which completely clarified this question.

More mysterious still was what Hilbert called his invariant energy equation, which he based on a complicated construct that came to be known as

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[1] For a detailed analysis of [Noether 1918b] and its slow reception by the mathematical community, see [Kosmann-Schwarzbach 2006/2011]. See also the commentary in [Siegmund-Schultze 2011].

[2] See [Alexandroff 1935], [Weyl 1935], [Dick 1981], [Kimberling 1981], and [Koreuber 2015].

Hilbert's energy vector e^l.[3] He derived this vector using classical techniques for producing differential invariants, an approach that differed sharply from Einstein's much more physically motivated derivation of energy conservation in [Einstein 1916a]. Klein later showed how Hilbert's energy vector arose naturally from the variational framework used in his theory [Klein 1918b]. Six years later, when Hilbert decided to publish a modified account of his original theory in [Hilbert 1924], he dropped all reference to his earlier approach to energy conservation, a clear indication that he no longer felt it had any importance for his unified field theory. Already in January 1918 Klein had exposed various hasty claims made in [Hilbert 1915]. His critique in [Klein 1918a] set the stage for Noether's insightful analysis that showed precisely how conservations laws and certain identities based on them arise in theories based on a variational principle. Klein took it upon himself to analyze the various proposals for conservation laws in differential form in [Klein 1918b]. In the course of doing so, he gave a simplified and much clearer derivation of Hilbert's invariant energy equation (4.7). He also succeeded in characterizing Einstein's formulation of energy conservation as presented in [Einstein 1918b]. Soon afterward, Klein took up Einstein's integral form for energy-momentum conservation in [Klein 1919].[4] In all of these studies he was assisted by Emmy Noether.

When Einstein began to study Hilbert's paper [Hilbert 1915] in earnest in May 1916, he naturally wondered whether there might be some deep connection between his own findings and Hilbert's energy equation. Hilbert thought this was probably the case, and he wrote as such to Einstein. He also informed him that he had already asked Emmy Noether to investigate this question, a circumstance that suggests he may have been disinclined to pursue this matter himself. What transpired afterward remains somewhat shrouded in mystery, but the present account will show that already by 1916 Noether had taken an important step toward solving this problem. In that year, she discovered that Hilbert's energy theorem as well as Einstein's formula for energy conservation shared a formal property closely connected with the field equations for gravitation. Although she never published on this topic, direct allusions to her discovery came to the surface in early 1918 when Klein and Hilbert published [Klein 1918a], an exchange of letters concerning the status of energy conservation in general relativity. Thanks to the recovery of a 9-page manuscript based on Noether's work, we can now reconstruct in outline the arguments she set forth in response to Hilbert's inquiry. In recounting this story, I have shifted the focus away from the immediate events of 1918 that led to Noether's seminal achievement, her paper [Noether 1918b]. In the course of its telling, however, it will become clear that the earlier events from 1916 – in particular Noether's findings with respect to energy conservation in

[3] As noted below, its definition (4.8) depends linearly on an arbitrary infinitesimal vector p^l, so e^l should be conceived as a vector field. For a detailed analysis of Hilbert's approach to conservation of energy-momentum, both in [Hilbert 1915] and in the earlier unpublished version in [Hilbert 2009], see [Renn/Stachel 2007].

[4] For a summary account of these issues as seen three years later, see [Pauli 1958, 175–178].

the theories of Hilbert and Einstein – directly presaged her later work, which arose from Klein's determination to clarify these issues.

2. On the Research Agendas of Hilbert and Klein

In the late spring of 1915, only shortly after Emmy Noether's arrival, Einstein came to Göttingen to deliver a series of six two-hour lectures on his new theory of gravitation, the general theory of relativity.[5] Einstein was pleased with the reception he was accorded, and expressed particular pleasure with Hilbert's reaction. "I am very enthusiastic about Hilbert," he wrote Arnold Sommerfeld, "an important man!" [Einstein 1998a, 147]. Hilbert had a long–standing interest in mathematical physics (see [Corry 2004], [Corry 2007]). Following Hermann Minkowski's lead, he and other Göttingen mathematicians felt strongly drawn to the formal elegance of relativity theory. For Hilbert, who had been advocating an axiomatic approach to physics for many years, relativity was ready-made for this program. In later years, he liked to joke that "physics had become too difficult for the physicists," a quip that was probably not intended all too seriously ([Weyl 1933, 347]). Although little is known about what transpired during the week of his visit to Göttingen, Einstein was clearly delighted by the response he received: "to my great joy, I succeeded in convincing Hilbert and Klein completely."[6] As for Hilbert's reaction to Einstein's visit, this encounter inspired him to consider whether general relativity might provide a fruitful framework for combining Einstein's gravitational theory with Gustav Mie's electromagnetic theory of matter.

By the fall of 1915, however, Einstein was no longer expressing the kind of self-satisfaction he felt immediately after delivering his Göttingen lectures. On 7 November, he wrote Hilbert: "I realized about four weeks ago that my methods of proof used until then were deceptive" [Einstein 1998a, 191]. Thus began a flurry of exchanges in which Einstein and Hilbert corresponded directly with one another as well as through Arnold Sommerfeld [Rowe 2001]. On November 20, Hilbert presented the first of his two communications to the Göttingen Scientific Society. Five days later, Einstein submitted [Einstein 1915], the last of his four notes on general relativity to the Berlin Academy. Abandoning the basic assumptions of his theory, he reaffirmed the centrality of general covariance while seeking a corresponding set of field equations for gravitation by making use of the Ricci tensor.[7]

The note [Einstein 1915] contains the fundamental equations:

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \tag{2.1}$$

[5]Einstein had been invited to Göttingen once before by Hilbert, in 1912, but declined that invitation (Albert Einstein to David Hilbert, 4 October 1912, in [Einstein 1995, 321–322].
[6]Einstein to W.J. de Haas, undated, probably August 1915, [Einstein 1998a, 162].
[7]On Einstein's struggles from this period, see [Stachel 1989], [Janssen 2014], [Janssen/Renn 2007], and [Janssen/Renn 2015].

where $g_{\mu\nu}$ and $R_{\mu\nu}$ are the metric and Ricci tensors, respectively. Einstein's argument for these equations was highly heuristic in nature, but he had already shown how, by using them in a simplified form, they could be used to calculate the displacement in Mercury's perihelion, a major breakthrough for precision measurements in solar astronomy. Hilbert, on the other hand, was able to derive gravitational field equations from a variational principle, an important mathematical achievement. Much as he had done in his other physical research, Hilbert hoped that by exploiting axiomatic and variational methods he would be able to place relativistic field theory on a firm footing.[8]

Initially, Emmy Noether worked closely with Hilbert, but she also assisted Felix Klein in preparing his lectures on the development of mathematics during the nineteenth century [Klein 1926]. Starting in the summer of 1916, Klein broke off these lectures in order to begin a 3-semester course on the mathematical foundations of relativity theory (published posthumously in [Klein 1927]). Much of what he presented during the first two semesters centered on the background to special relativity, including Maxwell's theory, but also the classical theory of algebraic and differential invariants. By the third semester, though, he entered the mathematical terrain of general relativity: Riemannian geometry and Ricci's absolute differential calculus. Klein's interests diverged rather strikingly from those of Minkowski and Hilbert, both of whom hoped to break new ground in electrodynamics. Unlike them, he was exclusively interested in the mathematical underpinnings of the new physics. Once Einstein pointed the way to a gravitational theory based on generalizing Minkowski space to a Riemannian manifold, Klein began to explore the purely mathematical foundations underlying Einstein's new Ansatz.[9] By the end of 1917, Klein sent Einstein a copy of the *Ausarbeitungen* of his lectures on the mathematical foundations of relativity. The latter's response was not very flattering: "it seems to me that you highly overrate the value of formal points of view. These may be valuable when an *already found* truth needs to be formulated in a final form, but they fail almost always as heuristic aids" [Einstein 1998a, 569].

Compared with Hilbert's research program, Klein's agenda was rather modest. Indeed, Hilbert was pursuing the far more ambitious goal of trying to find a connection between gravity and electromagnetism. His guiding ideas regarding the latter came from Gustav Mie's theory of matter.[10] Hilbert was especially attracted to Max Born's presentation of Mie's theory

[8] While some authors have portrayed the events of November 1915 as a race to arrive at the equations (2.1), recent research has made clear that this was indeed a major concern for Einstein, but much less so for Hilbert; see [Sauer 1999] and [Sauer/Majer 2009, 9–17].
[9] One year after Minkowski's premature death in 1909, Klein took up the connection between Minkowski's spacetime geometry, based on the invariance properties of the Lorentz group, and the ideas in his "Erlangen Program" [Klein 1910]. Klein's "Erlangen Program" [Klein 1872] was republished many times, e.g. in [Klein 1921-23, 460–497].
[10] Mie's approach to field physics also exerted a strong influence on Hermann Weyl up until around 1920. See Weyl's remarks in [Weyl 1918a, 1952: 211] and the note he later added on p. 216 after he became disillusioned with this program.

in [Born 1914][11] because of its mathematical elegance and reliance on variational methods. Variational principles had a longstanding place in classical mechanics, particularly due to the influential work of J. L. Lagrange, but their use in electrodynamics and field physics brought about numerous challenges. In the context of Mie's theory, Born showed how to derive its fundamental equations from a variational principle by varying the field variables rather than varying the coordinates for space and time.

Emmy Noether presumably had little knowledge of variational methods when she joined Hilbert's research group in 1915. What she knew very well, however, were related methods for using formal differential operators to generate algebraic and differential invariants.[12] In November 1915 she wrote to her friend and former Erlangen mentor, Ernst Fischer, to tell him about her work in Göttingen. Fischer had studied in Vienna under Franz Mertens, a leading expert on invariant theory whose work had influenced the young David Hilbert.[13] From Noether he now learned that Hilbert had created a buzz of excitement about invariant theory, so that even the physicist Gustav Hertz was studying the classical literature [Dick 1981, 30–31]. She herself had learned these older methods in Erlangen from Paul Gordan, who supervised her dissertation, a tedious study of the invariants and covariants associated with a ternary biquadratic form. Hertz was learning them from her *Doktorvater's* old lectures, edited by Georg Kerschensteiner in [Kerschensteiner 1885/1887]. Noether knew that Hilbert was pushing his team on with hopes for a breakthrough in physics, but she freely admitted that none of them had any idea what good their calculations might be [Dick 1981, 30–31].

No doubt Hilbert thought about this along lines first explored by Gustav Mie in his search for a suitable "world function" Φ that would lead to an electromagnetic theory of matter [Mie 1912]. Mie assumed that such a function Φ would have to be Lorentz covariant, thus compatible with the special theory of relativity, and furthermore that it should depend on the field variables alone. As Max Born pointed out, the latter assumption represented an important deviation from classical electron theory, in which the space and time coordinates enter the Lagrangian formalism. "In Mie's theory," he writes "the forces that hold atoms and electrons together should arise naturally from the formulation of Φ, whereas in the classical theory of electrons the forces have to be specifically added" [Born 1914, 753]. As for the demand that Φ be Lorentz covariant, Mie showed that this meant it had to be a function of just four invariant quantities.[14]

This same feature applied to Hilbert's world function H, which was invariant under general coordinate transformations. In his lecture course on foundations of physics from 1916/17, Hilbert emphasized the importance of

[11] For discussions of this paper, see [Corry 2004, 309–315] and [Smeenk/Martin 2007].
[12] For an introduction to this field, see [Olver 1999].
[13] Mertens influence on Hilbert is recounted in [Rowe 2018, 163–164].
[14] Several years later, it was discovered that Mie and his contemporaries had overlooked a fifth invariant; see [Smeenk/Martin 2007, 627, footnote 9].

restricting the possibilities for the Lagrangian H [Sauer/Majer 2009, 287–290]. He took this to be of the form $H = K + L$, where K is the Riemannian curvature scalar and the electromagnetic Lagrangian L depends on the metric tensor $g_{\mu\nu}$, but not on its derivatives. Hilbert noted that the $g_{\mu\nu}$ had to be present in L, as otherwise one could not construct any invariants from the electromagentic potentials alone. By the same token: "this assumption leads to a truly *powerful simplification*," since it means that L has to be a function of just four known invariants [Sauer/Majer 2009, 287]. Since the gravitational part of H was given by K, this meant that Hilbert's program rested on finding the requisite properties of these invariants in order to construct L.[15] His initial enthusiasm for these ideas did not last long, however, and by 1917 Hilbert's ambitions for a unified field theory of "everything" passed over to a ready acceptance of Einstein's position, namely that general relativity had no immediate relevance for microphysics [Renn/Stachel 2007].

Emmy Noether presumably gained some understanding of what Hilbert hoped to achieve for physics by drawing on advances in invariant theory. Yet if so, she surely never thought of her own work as motivated by Hilbert's physical program. In fact, she was already pursuing a program for invariant theory that was inspired by her collaboration with Ernst Fischer.[16] On 22 August 1917, she wrote him to announce that she had finally solved a problem that had occupied her attention since spring, namely the extension of a theorem proved by E.B. Christoffel and G. Ricci for quadratic differential forms to forms with any finite number of variables [Dick 1981, 33]. On 15 January 1918, Noether presented a lecture on her "Reduction Theorem" at a meeting of the Göttingen Mathematical Society, and ten days later Felix Klein submitted her paper [Noether 1918a] for publication. Drawing on methods in the calculus of variations introduced by Lagrange, Riemann, and Lipschitz, she shows how problems involving systems of differential invariants can be reduced to classical invariant theory, i.e. invariants of the projective group. Her treatment of Lagrangian derivatives as formal invariants reveals that this paper is closely related to the far more famous [Noether 1918b].

3. On Conservation Laws in General Relativity

Whereas Hilbert hoped to use Einstein's gravitational theory as a framework for a new unified field theory, Noether remained what she had always been: a pure mathematician. Her work thus aimed to clarify the mathematical underpinnings of general relativity, an effort strongly promoted by Felix Klein, who took up this challenge around the time that Hilbert's interests were turning back to the foundations of mathematics [Sauer/Majer 2009, 22]. In March of 1917, Klein initiated a correspondence with Einstein that sheds

[15] In this course from 1916/17, Hilbert made the additional assumption that the derivatives q_{hk} only enter L quadratically, from which he deduced that L takes the form $L = aQ + a_1Q_1 + a_2Q_2 + f(q)$, where Q, Q_1, Q_2, q are the four known invariants underlying his theory.

[16] For an idea of the scope of her research program in invariant theory, see [Noether 1923].

considerable light on how both thought about relativity theory and the general relationship between mathematical and physical reasoning. Their letters mainly reflect the three topics which were then uppermost in Klein's mind: 1) the conceptual links between relativity theory and his "Erlangen Program"; 2) the cosmological models proposed by Einstein and Willem de Sitter, in particular as these related to non-Euclidean geometries; and 3) the role of conservation laws in classical and relativistic physics. Only this last topic will be discussed here, but the others are suggestive of the broader range of issues central to the reception of general relativity in Göttingen.[17]

Beginning in March 1918, the correspondence between Klein and Einstein intensified markedly following the appearance of [Klein 1918a]. This paper arose from a presentation Klein made on 22 January 1918 to members of the Göttingen Mathematical Society, a talk that elicited a reaction from Hilbert one week later. The conclusions drawn from these two sessions were later summarized in the journal of the German Mathematical Society: "The 'conservation laws' valid for continua in classical mechanics (the impulse-energy theorems) are already contained in the field equations in Einstein's newly inaugurated theory; they thereby lose their independent significance."[18]

Klein and Hilbert afterward agreed to publish their respective viewpoints in the *Göttinger Nachrichten* as an epistolary exchange.[19] Considering that they both lived only a short distance from one another on the Wilhelm Weber Strasse, one might wonder why they chose to publish the gist of their discussions as an exchange of correspondence. In any event, the views they set forth harmonized and were surely meant to be seen as representing the consensus opinion on these matters in Göttingen. In [Klein 1918a], Klein underscored that Hilbert's invariant energy equation should not be viewed as a conservation law in the sense of classical mechanics. The latter could only be derived by invoking physical properties of matter, whereas Hilbert's equation followed directly from the gravitational field equations by means of purely formal considerations. Klein further remarked that Emmy Noether had already noticed this and had worked out all the details in a manuscript, a text she had shown him. "You know," he wrote, "that Miss Noether advises me continually regarding my work, and that, in fact, it is only thanks to her that I have understood these questions" [Klein 1918a, 559].

Hilbert was certainly very well aware of this, and he responded as follows:

> I fully agree with the substance of your statements on the energy theorems. Emmy Noether, on whom I have called for assistance more than a year ago to clarify this type of analytical question concerning my energy theorem, found at that time that the energy

[17] For a discussion of topic 2), see [Rowe 2018, 279–299].

[18] *Jahresbericht der Deutschen Mathematiker–Vereinigung*, 27 (1918), ("Mitteilungen und Nachrichten"), p. 28.

[19] Klein had already presented a preliminary version of [Klein 1918a] at a meeting of the scientific society on 25 January.

> components that I had proposed – as well as those of Einstein – could be formally transformed, using the Lagrangian differential equations . . . of my first note, into expressions whose divergence vanishes identically . . .. [Klein 1918a, 560–561].

As it happens, a Swiss student named Rudolf Jakob Humm attended Klein's lecture and was impressed by what he heard. Humm was studying relativity under Hilbert, but he had also spent a semester in Berlin attending Einstein's lecture course. Since he was particularly interested in energy conservation, Humm surely read [Klein 1918a], which would have made him aware of Noether's manuscript had he not known about it already before. In any event, Humm must have approached her at some point to ask if he could copy part of this text.[20]

Humm grew up in Modena and later completed his secondary education at the Kantonsschule in Aarau. This was the same institution Einstein had attended for one year before he entered the Polytechnicum in Zürich, a circumstance that possibly inspired Humm to study relativity in Germany. He first studied mathematics in Munich in 1915, before moving on to Göttingen. By the winter semester of 1916/17 he was thoroughly steeped in theoretical physics.

During wartime, university enrollments plummeted, so Humm was moving in a small world in which people saw one another nearly every day. His contacts with Emmy Noether were apparently rather fleeting, whereas he regularly interacted with several fellow natives of Switzerland, including the physicist Paul Scherrer and his wife, Paul Finsler, and Richard Bär. Humm also socialized with Vsevolod Frederiks, one of several Russians studying physics and mathematics in Göttingen [Rowe 2004, 114–115], and he befriended Willy Windau, a blind mathematician who went on to take his doctorate under Hilbert in 1920. Another sometime companion was the astronomer Walter Baade, who took his degree in 1919.

One evening in April 1917, he and Baade met for drinks at the Hotel National. Humm was somewhat despondent on this occasion, in part because of the meager course offerings for the coming semester. He had been following Hilbert's relativity course with enthusiasm, but for the summer, the master would be teaching only a four-hour course on set theory. Over the course of that evening, Baade convinced him to leave Göttingen for Berlin, where Einstein had already begun teaching a course on relativity. A few days later, Humm was already settled in and looking forward to Einstein's course, which was held on Thursdays from 2 to 4. He also made plans to attend the physics colloquium, which was run by Heinrich Rubens.

Humm had missed the first two lectures, so he had some questions after hearing the third. Evidently, Einstein offered to meet with him the following Saturday, an encounter that led to a series of remarks by the physicist that

[20] This copy, written in Humm's hand, is just one of the many documents in his posthumous papers relating to his interest in general relativity during the war years (Nachlass Rudolf Jakob Humm, Zentralbibliothek Zürich).

Humm tried to reconstruct in his diary. Einstein had recently read Hilbert's second note on the foundations of physics [Hilbert 1917]. There, Hilbert had introduced a special coordinate system in order to preserve causal relations in general relativity, but Einstein thought this was inadmissible because it could lead to worldlines that converge, thereby yielding space-time singularities. He had already mentioned this criticism two weeks earlier in a letter to Felix Klein.[21]

This was only one of several conversations Humm had with Einstein during his three-month stay in Berlin. Alongside Einstein's course, he also attended Max Planck's lectures on quantum theory as well as Rubens's weekly colloquium, which met on Wednesdays. He found this all quite stimulating, but he also missed the conveniences of Göttingen's Lesezimmer. In Berlin one had to order books from the library, so there was no opportunity to browse open shelves to pick out the volumes one might want to read. Rubens had asked Humm to speak in the colloquium in early August, but this plan was aborted after Einstein fell ill in mid-July. His assistant, Jakob Grommer, then took over the course, while Einstein left for Switzerland to recover from an intestinal ailment. For Humm, this sudden turn of events meant that he had little incentive to stay in Berlin any longer. So he canceled his colloquium lecture and soon thereafter returned to Göttingen.

Humm was surely well-versed when it came to the various proposals for dealing with energy conservation in general relativity. He had attended Hilbert's year-long course on "Die Grundlagen der Physik", in which this topic received renewed attention [Sauer/Majer 2009, 304–306]. When he arrived back in Göttingen for the winter semester 1917/18, Hilbert appointed him to prepare the official *Ausarbeitung* for his lecture course on electron theory. At the end of Einstein's final lecture, Humm was keen to learn his opinion about one of the most controversial parts of his theory, namely Einstein's pseudo-tensor for representing gravitational energy. Throughout 1917 and 1918, Einstein argued against much skeptical opinion that the expression for gravitational energy could not be a general tensor; on the contrary, it needed to vary with the coordinate frame. Humm recorded this response in his notes from Einstein's lecture:

> I asked Einstein if it would be possible to generalize the conservation equation
> $$\frac{\partial(\mathfrak{T}_\mu^\sigma + \mathfrak{t}_\mu^\sigma)}{\partial x_\sigma} = 0$$
> so that it would contain only real tensors. He thought not: one does not shy from writing
> $$\frac{\partial(T+U)}{\partial t} = 0.$$
> in classical mechanics, where U in an invariant under Galilean transformations, but T is not. So it not so terrible to have the

[21] Einstein to Klein, 24 April 1917, [Einstein 1998a, 426].

general tensor $\mathfrak{T}_\mu^\sigma$ next to the special $\mathfrak{t}_\mu^\sigma$. If one considers an accelerative field, then there will be a $\mathfrak{t}_\mu^\sigma$, even though the field can be transformed away. In the end, one can operate with any arbitrary concept, and it cannot be said that they have to be tensor quantities; the [Christoffel symbols] are also not tensors, but one operates with them. The $\mathfrak{t}_\mu^\sigma$ are the quantities that deliver the most. (Nachlass Rudolf Jakob Humm, Zentralbibliothek Zürich)

Humm was strongly drawn to Einstein's highly conceptual way of thinking about fundamental physical problems, an approach he contrasted with Hilbert's purely mathematical approach. Energy conservation and the equations of motion in general relativity would thenceforth become his principal research agenda.

Meanwhile, Humm continued to stay in contact with Einstein, who submitted two of his papers for publication in *Annalen der Physik*. The first of these was [Humm 1918], written in May 1918, just two months before Emmy Noether presented her paper [Noether 1918b]. In it, Humm takes considerable care to explain how one can apply different variational methods to obtain results adapted to a particular physical setting. Among his findings, he could show that Einstein's differential equations for conservation of energy were derivable from the equations of motion, i.e., the assumption that a test particle moves along a geodesic in curved space-time. Humm submitted [Humm 1919], his second paper, just one month after Noether completed hers; again, one finds striking parallels between them. This second contribution aimed to show that Einstein's energy equations could be seen as equivalent to equations of motion, an argument based on certain analogies with Lagrangian mechanics.

Humm's transcription of Noether's results from 1916 will be discussed below. First, however, it will be necessary to consider how Hilbert derived his invariant energy vector in [Hilbert 1915], after which I will briefly describe Einstein's handling of energy conservation in general relativity in [Einstein 1916a].

4. Hilbert's Approach to Energy in [Hilbert 1915]

When Einstein delivered his six Wolfskehl lectures in Göttingen in the late spring of 1915, he was advancing a version of a gravitational theory that he had earlier worked out with the help of the mathematician Marcel Grossmann [Einstein/Grossmann 1913]. At this time, he was convinced that if such a theory were based entirely on the principle of general covariance, then it would necessarily be undetermined. For this reason, he and Grossmann reached the conclusion that the gravitational field equations could not be generally covariant. Instead, their equations were covariant only with respect to a more restricted group that included the linear transformations. In Göttingen Einstein quite possibly spoke about the possibility of using energy conservation in order to bring about this restriction, one of several problems he had yet to solve.

In any event, Hilbert's initial attempt to subsume Mie's theory within the context of general relativity followed Einstein's then current belief that the field equations themselves could not be generally covariant. Moreover, to avoid this problem he struck on the idea of utilizing energy conservation to restrict the system of allowable coordinates, a method designed to preserve causal relations. This initial foray into general relativity, however, never found its way into print. Hilbert's contemporaries were therefore unaware that his original approach to energy conservation differed fundamentally from the one that appeared in his published note, [Hilbert 1915]. The discrepancy was only discovered in the late 1990s when historians discovered that, although this note still bore the original date of submission (20 November 1915), Hilbert had heavily revised it after receiving the page proofs in December 1915 [Corry/Renn/Stachel 1997].[22]

In these page proofs, published as [Hilbert 2009], Hilbert introduces a linear invariant as the "energy form," from which he derives four coordinate conditions from a divergence equation. He then introduces this coordinate system as an "axiom for space and time," thereby obtaining a total of 14 differential equations for the 14 field variables required for combining Einstein's and Mie's theories. He adopted this strategy in order to circumvent the problem he foresaw if the theory only admitted 10 equations, thereby leaving four degrees of freedom for the motion of a physical system. Hilbert dropped all this, however, in the published version of [Hilbert 1915], where his modified energy law is fully covariant. Nevertheless, as described in [Brading and Ryckman 2018], he continued to struggle with the problem of reconciling general covariance with causality, which returns to the fore in [Hilbert 1917], the sequel to his first note on "Die Grundlagen der Physik". As John Stachel notes in [Stachel 1992], this paper was the first attempt to deal with the Cauchy problem in general relativity. In it, Hilbert commented:

> As far as the causality principle is concerned, if the physical quantities and their time derivatives are known in the present in any given coordinate system, then a statement will only have physical meaning if it is invariant with respect to those transformations for which the coordinates used are precisely those for which the known present values remain invariant. I claim that all assertions of this kind are uniquely determined for the future as well, i.e., that the causality principle is valid in the following formulation: From knowledge of the fourteen potentials ... in the present all statements about them in the future follow necessarily and uniquely insofar as they have physical meaning. [Hilbert 1917, 61]

[22] The extant page proofs are incomplete, however. What they likely once contained has been discussed in [Sauer 2005] and in [Renn/Stachel 2007].

Hilbert's theory in [Hilbert 1915] was based on two axioms that concern the properties of a "world function"

$$H(g_{\mu\nu}, g_{\mu\nu,l}, g_{\mu\nu,lk}, q_s, q_{s,l}).$$

This H is taken to be a scalar-valued function that does not depend explicitly on the spacetime coordinates w_s but rather on the ten components of the symmetric metric tensor $g_{\mu\nu}$ and its first and second derivatives as well as four electromagnetic potentials q_s and their first derivatives. Hilbert notes that H could just as well be defined by means of the contravariant arguments $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{lk}^{\mu\nu}$, which he adopts afterward. Axiom I then asserts that under infinitesimal variations of the field functions $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$ and $q_s \to q_s + \delta q_s$,

$$\delta \int H\sqrt{g}d\omega = 0,$$

where $g = |g^{\mu\nu}|$ and $d\omega = dw_1 dw_2 dw_3 dw_4$. This variational principle is understood to apply throughout a finite region of space-time. Axiom II then simply states that this world function H is taken to be invariant under general coordinate transformations.

By virtue of Axiom I, Hilbert obtained ten Lagrangian differential equations for the ten gravitational potentials $g^{\mu\nu}$:

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} = 0. \tag{4.1}$$

Similarly, for the four electrodynamic potentials q_s, he derived the four equations:

$$\frac{\partial\sqrt{g}H}{\partial q_h} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial q_{hk}} = 0. \tag{4.2}$$

Hilbert called the first set of equations the fundamental equations of gravitation and the second the fundamental equations of electrodynamics, abbreviating these to read:

$$[\sqrt{g}H]_{\mu\nu} = 0, \quad [\sqrt{g}H]_h = 0. \tag{4.3}$$

In the course of developing his theory, Hilbert focused on the special case where the Lagrangian H takes the form $H = K + L$. Here K is the curvature scalar obtained by contracting the Ricci tensor $K_{\mu\nu}$, i.e., $K = \sum_{\mu\nu} g^{\mu\nu}K_{\mu\nu}$. He placed no special conditions on the Lagrangian L, but noted that it contained no derivatives of the metric tensor, so that $H = K + L(g^{\mu\nu}, q_s, q_{s,l})$. Utilizing a general theorem for constructing differential invariants from a given invariant, Hilbert showed how L led to a differential equation that served as a generalized Maxwell equation, as in Gustav Mie's electromagnetic theory of matter.

Hilbert's central claim concerned four independent linear combinations satisfied by the $[\sqrt{g}L]_h$ and their derivatives. He showed toward the end of [Hilbert 1915] how these can be derived from the fundamental equations of gravitation. His argument depended on first identifying the electromagnetic part of his energy vector e^l with Mie's expression for energy. Hilbert's more general expression passed over to Mie's when the metric tensor $g_{\mu\nu}$ took on values for a flat spacetime. He then showed that this same expression could be linked to the gravitational field equations, which for $H = K + L$ take the form:

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial \sqrt{g}L}{\partial g^{\mu\nu}} = 0.^{23} \tag{4.4}$$

The electromagnetic part of e^l could then be written:

$$\frac{-2}{\sqrt{g}} \sum_{\mu,s} \frac{\partial \sqrt{g}L}{\partial g^{\mu s}} g^{\mu l} p^s. \tag{4.5}$$

After carrying out a number of quite complicated transformations and making use of (4.4), Hilbert obtained four identities involving the Lagrangian expressions $[\sqrt{g}L]_m$ and their first derivatives:

$$\sum_m (M_{\mu\nu}[\sqrt{g}L]_m + q_v \frac{\partial [\sqrt{g}L]_m}{\partial w_m}) = 0,^{24} \tag{4.6}$$

where $M_{\mu\nu} = q_{\mu\nu} - q_{\nu\mu}$.

Somewhat misleadingly, however, he related this specific result to a much more general one, announced but not proved at the outset. This was his Theorem I, which generalized the situation described by axioms I and II to any invariant J in n variables and their derivatives. From this invariant variational framework one can then derive n Lagrangian differential equations, from which $n - 4$ of these lead to four identities satisfied by the other four and their total derivatives. So stated, this theorem asserts that four of the fourteen equations $[\sqrt{g}H]_{\mu\nu} = 0$ $[\sqrt{g}H]_h = 0$ can be deduced directly from the other ten. Hilbert seized on this result to make a strong physical claim:

> ...on account of that theorem we can immediately make the assertion, *that in the sense indicated the electrodynamic phenomena are the effects of gravitation.* In recognizing this, I discern the simple and very surprising solution of the problem of Riemann, who was the first to search for a theoretical connection between gravitation and light. [Hilbert 1915, 397–398][25]

Hilbert would later drop this passage in [Hilbert 1924], a new version of his two notes, although the theorem he stated was correct and certainly important.

A more immediately controversial and confusing aspect, however, concerned Hilbert's handling of energy conservation. In the first part of his paper,

[23]Much has written about how Hilbert came to recognize the Lagrangian derivative $[\sqrt{g}K]_{\mu\nu}$ as being identical with the Einstein tensor $\sqrt{g}(K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K)$, but it should not be overlooked that this claim plays no role whatsoever in the arguments presented in [Hilbert 1915]. Hilbert surely felt it important to establish this linkage with Einstein's theory of gravitation, but the results he set forth did not make use of the specific form of Einstein's field equations.

[24]His derivation of this equation is found on [Hilbert 1915, 405–406].

[25]Hilbert was alluding here to Riemann's posthumously published "Gravitation und Licht," in [Riemann 1876, 496].

he constructed a complicated invariant e^l, his energy vector, from which he proved that its divergence vanished; this was his invariant energy theorem:

$$\sum_l \frac{\partial \sqrt{g} e^l}{\partial w_l} = 0. \tag{4.7}$$

The energy vector e^l is defined by starting with an arbitrary vector p^l and then building four other vectors by means of differential invariants. The resulting construction takes this form:

$$e^l = H\, p^l - a^l - b^l - c^l - d^l. \tag{4.8}$$

Each of these five terms is an invariant, but only the first depends on both the gravitational and electromagnetic potentials. The vectors a^l, b^l contain expressions without the q_s, whereas the c^l, d^l are independent of the $g^{\mu\nu}$. Hilbert emphasized that his energy equation holds for any H satisfying the first two axioms, even though the construction of e^l clearly reveals that he had the special case $H = K + L$ in mind. Thus, while he formulates Theorem II for a general invariant J of the type H, Hilbert decomposes the operator P that acts as the first polar:

$$\sum_l \frac{\partial J}{\partial w_s} p^s = P(J)^{26} \tag{4.9}$$

by writing $P = P_g + P_q$ in order to separate the gravitational and electromagnetic terms, where

$$P_g = \sum_{\mu,\nu,l,k} (p^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} + p^{\mu\nu}_l \frac{\partial}{\partial g^{\mu\nu}_l} + p^{\mu\nu}_{lk} \frac{\partial}{\partial g^{\mu\nu}_{lk}})$$

and

$$P_q = \sum_{l,k} (p_l \frac{\partial}{\partial q_l} + p_{lk} \frac{\partial}{\partial q_{lk}}),$$

where the lower suffixes in p denote coordinate derivatives.

Hilbert then applies the first operator to polarize the expression $\sqrt{g}H$:

$$P_g(\sqrt{g}H) = \sum_{\mu,\nu,l,k} (p^{\mu\nu} \frac{\partial \sqrt{g}H}{\partial g^{\mu\nu}} + p^{\mu\nu}_l \frac{\partial \sqrt{g}H}{\partial g^{\mu\nu}_l} + p^{\mu\nu}_{lk} \frac{\partial \sqrt{g}H}{\partial g^{\mu\nu}_{lk}}). \tag{4.10}$$

Using formal properties of tensors, Hilbert introduces the two vectors a^l, b^l and shows that they satisfy the equation

$$P_g(\sqrt{g}H) - \sum_l \frac{\partial \sqrt{g}(a^l + b^l)}{\partial w_l} = \sum_{\mu,\nu} [\sqrt{g}H]_{\mu\nu} p^{\mu\nu}.$$

[26]This equation vexed Einstein, who wrote to Hilbert on 25 May 1916 (see below). Hilbert noted in his reply that the coefficients in the power series expansion arising from a displacement of the variables in the invariant J will themselves be invariant, and furthermore that the first order terms are those given by $P = P_g + P_q$.

In constructing the vector a^l, he first notes that the coefficient of $p^{\mu\nu}_{lk}$ in the expression for (4.10), namely $\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}_{lk}}$, is a mixed fourth-order tensor, which enables him to produce a^l by multiplying this tensor with another of the third rank. Emmy Noether would point out in 1916 that the second derivatives of the metric tensor that appear in the definition of a^l cannot be eliminated by means of the field equations; she took this as an indication that Hilbert's energy vector was not analogous to a first integral in classical mechanics.

By an analogous argument using the second operator, Hilbert obtains the vector c^l which satifies the equation:

$$P_q(\sqrt{g}H) - \sum_l \frac{\partial\sqrt{g}(c^l)}{\partial w_l} = \sum_k [\sqrt{g}H]_{\mu\nu} p_k.$$

Adding these two equations and applying the fundamental field equations (4.3), it follows that

$$P(\sqrt{g}H) = \sum_l \frac{\partial\sqrt{g}(a^l + b^l + c^l)}{\partial w_l}.$$

Hilbert now applies the identity (4.9) to this equation to obtain

$$P(\sqrt{g}H) = \sum_s \frac{\partial\sqrt{g}Hp^s}{\partial w_s},$$

which leads immediately to the divergence equation:

$$\frac{\partial}{\partial w_l}\sqrt{g}(Hp^l - a^l - b^l - c^l) = 0.$$

To complete the construction of the energy vector (4.8), Hilbert defined d^l by making use of the skew symmetric tensor $\frac{\partial H}{\partial q_{lk}} - \frac{\partial H}{\partial q_{kl}}$. Since this d^l has vanishing divergence, it follows immediately that the vector e^l does as well, which completes his proof of (4.7).

Hilbert wrote at the outset of this derivation that this was a fundamental result for his theory and that it followed from his two axioms alone, though he of course also made use of Theorem II. His readers must have been quite mystified, however, by the fact that he derived another result, Theorem III, before taking up his energy theorem.[27] This theorem plays no role in Hilbert's treatment of energy conservation but, as we shall see below, it forms the starting point for Emmy Noether's analysis of Hilbert's energy vector.

5. Einstein's Approach to Energy Conservation

As is well known, Einstein originally introduced gravitational effects into his special theory of relativity (SR) by means of the equivalence principle. Once he accepted Minkowski's approach to SR, he eventually found a way to adapt

[27] Hilbert used Theorem III to show how electromagnetic energy (4.5), expressed in terms of the derivatives of L with respect to the gravitational potentials $g^{\mu\nu}$, leads by virtue of the gravitational equations (4.4) to the identities (4.6).

the equivalence principle to it. In SR, force-free motion in an inertial frame of reference takes place along a straight-line path with constant velocity. Viewed from a non-inertial frame, on the other hand, this path of motion will be a geodesic curve in a flat spacetime

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}, \tag{5.1}$$

since this equation is independent of the coordinate system. Einstein made the plausible assumption that this geodesic motion also holds in the non-flat case, i.e. in a spacetime region for which it is impossible to find a coordinate system that leads to the Minkowski metric in SR.[28] This geometrical assumption served as the starting point for his gravitational theory; afterward it stood as a sturdy bridge that joined the special and general theories of relativity.

Einstein's classic paper [Einstein 1916a] was published as a separate brochure that came out just before Einstein began an interesting correspondence with Hilbert, which will be discussed below. In [Einstein 1916a] one encounters a number of arguments leading to different formulations of the gravitational field equations. From the outset, Einstein posed the unimodular coordinate condition $\sqrt{-g} = 1$,[29] which leads to a significant simplification of the field equations (2.1). He first considered the matter-free case ($T_{\mu\nu} = 0$), $R_{\mu\nu} = 0$. Here $R_{\mu\nu}$ is the Ricci tensor, which simplifies in unimodular coordinates, so the ten differential field equations can be written

$$\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = 0, \tag{5.2}$$

where the $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols of the second kind (another popular notation is $\Gamma^\sigma_{\mu\lambda} = \left\{ {\sigma \atop \mu\lambda} \right\} = -\left\{ {\mu\lambda \atop \sigma} \right\}$).

Einstein derives another form of the equations (5.2) by using variational methods. Assuming $\sqrt{-g} = 1$, he takes the scalar

$$H = \sum_{\alpha\beta\mu\nu} g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \tag{5.3}$$

and writes

$$\delta\left\{\int H d\tau\right\} = 0.$$

Carrying out the variation yields field equations in the form:

$$\frac{\partial}{\partial x_\alpha}\left\{\frac{\partial H}{\partial g^{\mu\nu}_\alpha}\right\} - \frac{\partial H}{\partial g^{\mu\nu}} = 0. \tag{5.4}$$

After a series of intermediate calculations, he obtains:

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[28] A number of investigators, including Hermann Weyl, afterward showed how the geodesic equation for motion could be deduced from the field equations (see [Havas 1989]). Einstein, however, was reluctant to follow this lead for reasons discussed in [Kennefick 2005] and [Lehmkuhl 2017].

[29] On Einstein's use of unimodular coordinates, see the discussion in [Janssen/Renn 2007].

$$\sum_{\alpha} \frac{\partial t_{\sigma}^{\alpha}}{\partial x_{\alpha}} = 0; \ -2\chi t_{\sigma}^{\alpha} = \sum_{\mu\nu} \left\{ g_{\sigma}^{\mu\nu} \frac{\partial H}{\partial g_{\alpha}^{\mu\nu}} \right\} - \delta_{\sigma}^{\alpha} H. \tag{5.5}$$

Einstein noted that although t_{σ}^{α} is not a general tensor, the equations (5.5) are valid whenever $\sqrt{-g} = 1$. He interpreted the t_{σ}^{α} pseudo-tensor as representing the energy components of the gravitational field and (5.5) as expressing the equation for conservation of momentum and energy in the vacuum case. For the pseudo-tensor, he derives the equation

$$\chi t_{\sigma}^{\alpha} = \sum_{\mu\nu\lambda} \frac{1}{2} \delta_{\sigma}^{\alpha} g^{\mu\nu} \Gamma_{\mu\beta}^{\lambda} \Gamma_{\nu\lambda}^{\beta} - g^{\mu\nu} \Gamma_{\mu\beta}^{\alpha} \Gamma_{\nu\alpha}^{\beta}. \tag{5.6}$$

Einstein's generalization of (5.5) in the presence of a matter tensor T_{μ}^{σ} takes the form

$$\frac{\partial (T_{\mu}^{\sigma} + t_{\mu}^{\sigma})}{\partial x_{\sigma}} = 0. \tag{5.7}$$

He obtains this by deriving yet another form for the field equations (5.2), still assuming the condition $\sqrt{-g} = 1$:

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_{\alpha}} \left(g^{\sigma\beta} \Gamma_{\mu\beta}^{\alpha} \right) = -\chi \left(t_{\mu}^{\sigma} - \frac{1}{2} \delta_{\mu}^{\sigma} t \right), \tag{5.8}$$

where $t = \sum_{\alpha} t_{\alpha}^{\alpha}$.

He then modifies equations (5.8) by replacing t_{μ}^{σ} with $t_{\mu}^{\sigma} + T_{\mu}^{\sigma}$ to obtain:

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_{\alpha}} \left(g^{\sigma\beta} \Gamma_{\mu\beta}^{\alpha} \right) = -\chi \left[(t_{\mu}^{\sigma} + T_{\mu}^{\sigma}) - \frac{1}{2} \delta_{\mu}^{\sigma} (t + T) \right]. \tag{5.9}$$

By means of (5.9) and some intermediate calculations, Einstein derives the differential form for conservation of momentum and energy (5.7). These results from [Einstein 1916a] clearly differ sharply from the findings in [Hilbert 1915] discussed above. Nevertheless, Emmy Noether was able to show that Hilbert's e^{l} and Einstein's t_{σ}^{α} both possessed a common property which seemed to reflect that the energy laws in general relativity differ from those in classical mechanics or special relativity.

Soon after he published [Einstein 1916a] in May 1916, Einstein made a conscientious attempt to understand how Hilbert developed his far more complicated approach to energy-momentum conservation published in [Hilbert 1915].[30] Einstein struggled to understand the arguments in Hilbert's first note as he prepared to speak about it in Heinrich Ruben's colloquium. Twice he turned to Hilbert for clarifications, writing: "I admire your method, as far as I have understood it. But at certain points I cannot progress and therefore ask that you assist me with brief instructions" (Einstein to Hilbert, 25 May 1916,

[30] The complications were in part due to the fact that Hilbert decided to alter his definition of energy in the page proofs of his original submission from 20 November 1915. Thus the version in [Hilbert 1915] actually reflects an important shift in Hilbert's understanding of this aspect of his theory. For details, see Tilman Sauer's commentary in [Sauer/Majer 2009], pp. 11–13.

[Einstein 1998a, 289]). He was particularly baffled by Hilbert's energy theorem, admitting that he could not comprehend it at all – not even what it asserted.[31]

Hilbert wrote back just two days later. He easily explained how, via the operation of polarization, an invariant J will lead to a new invariant $P(J)$, its first polar. He then went on to say:

> My energy law is probably related to yours; I have already assigned this question to Miss Noether. As concerns your objection, however, you must consider that in the boundary case $g^{\mu\nu} = 0, 1$ the vectors a^l, b^l by no means vanish, as K is linear in the $g^{\mu\nu}_{\sigma\kappa}$ terms and is differentiated with respect to these quantities. For brevity I give you the enclosed paper from Miss Noether.

Hilbert's conjecture regarding the relationship between his and Einstein's versions of energy conservation was surely no more than a first guess. Even on the purely formal level, he could hardly assert that his energy vector e^l stood in some obvious relation to Einstein's pseudo-tensor t^α_σ.

Einstein was well aware that Noether was working closely with Hilbert and that the latter had been trying to break the resistance in the faculty to her appointment as a *Privatdozent*. Despite strong support by the members of the natural sciences division, however, all such efforts proved impossible during wartime. Only after the fall of the German Reich and the advent of the Weimar Republic did these efforts succeed (see [Tollmien 1990]). Einstein responded to Hilbert's letter shortly afterward:

> Your explanation of equation [(4.9)] in your paper delighted me. Why do you make it so hard for poor mortals by withholding the technique behind your ideas? It surely does not suffice for the thoughtful reader if, although able to verify the correctness of the equations, he cannot have a clear view of the overall plan of the analysis.

Einstein was far more blunt about this in a letter he wrote to Paul Ehrenfest on May 24: "Hilbert's description doesn't appeal to me. It is unnecessarily specialized regarding 'matter,' is unnecessarily complicated, and not straightforward (= Gauss-like) in set-up (feigning the super-human through camouflaging the methods)" [Einstein 1998a, 288]. After receiving Hilbert's explanations, he may have felt somewhat more conciliatory. Certainly he made every effort to understand Hilbert's arguments, and could report: "In your paper everything is understandable to me now except for the energy theorem. Please do not be angry with me that I ask you about this again" [Einstein 1998a, 293]. After explaining the difficulty he still had, Einstein ended by writing that it would suffice if Hilbert asked Emmy Noether to

[31] Hilbert claimed not only that the energy vector e^l depended solely on the metric tensor and its derivatives, he also showed that by passing to a flat metric its electromagnetic part turned out to be closely related to a formulation for energy derived from Mie's theory. Einstein was puzzled about this derivation, since the argument seemed to show that not only the divergence of the energy term but this term itself would have to vanish.

clarify the point that was troubling him. This turned out to be a quite trivial matter, so Hilbert answered Einstein directly. The latter then responded with thanks, adding that "now your entire fine analysis is clear to me, also with respect to the heuristics. Our results are in complete agreement" [Einstein 1998a, 295].

What Einstein meant by this would seem quite obscure. Perhaps he only meant to assure Hilbert that he would no longer be pestering him about these matters. One must assume that Hilbert had just as little interest to enter these waters further, for how else to account for the fact that he failed to publicize Emmy Noether's findings, which clearly stemmed from this correspondence with Einstein? Not until Felix Klein began to take an interest in the status of conservation theorems in GR more than a year later did Noether's name receive any attention in this connection.

6. On Noether's Unpublished Manuscript from 1916

Noether's original manuscript no longer survives, but fortunately R. J. Humm made a partial transcription, probably in early 1918. He also included the original pagination, which indicates that his manuscript begins with page 15 of her text. Since a number of steps in Noether's arguments are based on equations from the first 14 pages, any attempt to reconstruct how she obtained these results would be necessarily conjectural. Here I will simply take such claims as established facts; I will follow the same procedure when Noether draws on results in [Hilbert 1915] and [Einstein 1916a]. By so doing, the general train of her arguments is not difficult to follow. They show that Hilbert's energy vector as well as Einstein's pseudotensor representing gravitational energy can both be decomposed into two parts, one of which will have vanishing divergence, whereas the other vanishes as a result of the field equations. Her analysis draws closely on Hilbert's own techniques in [Hilbert 1915], which she then applies in order to analyze Einstein's construct in [Einstein 1916a].

Noether's analysis of Hilbert's energy vector exploited the fact that his "world function" takes the form $H = K + L$ and that K is defined solely by the metric tensor and its first and second derivatives. In her manuscript, she employs notation that deviates only slightly from that found in the two papers she discusses. For [Hilbert 1915] she begins her discussion of Hilbert's energy vector (4.8) by looking at the vacuum case, $H = K$, where the last two terms $c^l = d^l = 0$, since these only enter through the electromagnetic potential. She proceeds then to produce a decomposition of Hilbert's expression into a sum of two vectors, one of which vanishes by virtue of the field equations, whereas the divergence of the other vanishes identically, i.e., independent of the field equations.

Hilbert writes p^i_s for $\frac{\partial p^i}{\partial w_s}$, and for the Lie variation:

$$\delta g^{\mu\nu} \equiv p^{\mu\nu} = \sum_s (g^{\mu\nu}_s p^s - g^{\mu s} p^\nu_s - g^{\nu s} p^\mu_s). \tag{6.1}$$

Noether follows Hilbert's Theorem III, writing:

$$i_s = \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} g_s^{\mu\nu} \tag{6.2}$$

$$i_s^l = -2 \sum_{\mu} [\sqrt{g}K]_{\mu s} g^{\mu l}, \tag{6.3}$$

and then noting that

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_{sl} i_s\, p^s + i_s^l\, p_l^s. \tag{6.4}$$

Hilbert's Theorem III asserts that $i_s = \sum_l \frac{\partial i_s^l}{\partial w_l}$,[32] which means that i_s can be written as a divergence or expressed in the form of the identity:

$$\sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} g_s^{\mu\nu} + 2 \sum_l \frac{\partial([\sqrt{g}K]_{\mu s} g^{\mu l})}{\partial w_l} = 0. \tag{6.5}$$

Noether exploits this in showing that the left side of (6.4) can be written as a divergence. To do this she introduces the vector

$$i^l = \sum_s \frac{i_s^l}{\sqrt{g}} p^s, \tag{6.6}$$

in order to rewrite equation (6.4) as

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_{sl} (\frac{\partial i_s^l}{\partial w_l}\, p^s + i_s^l\, p_l^s) = Div(\sum_s \frac{i_s^l}{\sqrt{g}}\, p^s) = Div(i^l). \tag{6.7}$$

Drawing on previous calculations, she asserts that for $e^l = K\, p^l - a^l - b^l$

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu} = Div(e^l). \tag{6.8}$$

It follows from equations (6.7) and (6.8) that $Div(e^l) = Div(i^l)$ and furthermore that $Div(e^l - i^l) = 0$ holds identically. By virtue of the fundamental equations $[\sqrt{g}K]_{\mu\nu} = 0 = Div(i^l)$ for arbitrary p^s, whereas (6.3) shows that i_s^l also vanishes, which means that by definition (6.6) $i^l = 0$.

From this, Noether concludes that in the vacuum case one can always decompose Hilbert's energy vector as:

$$e^l = i^l + (e^l - i^l), \tag{6.9}$$

where the first part vanishes as a consequence of the fundamental equations $[\sqrt{g}H]_{\mu\nu} = 0$, whereas the divergence of the second part vanishes identically. She then summarizes the physical significance of this result as follows: "The energy is probably *not* to be regarded as a first integral (as in classical

[32] [Renn/Stachel 2007, 895] note that Theorem III "corresponds to the contracted Bianchi identities," an insight that Hilbert and his contemporaries failed to notice, although the Bianchi identities had been discovered decades earlier. For the story of their recovery in the context of general relativity, see [Rowe 2018, 263–272].

mechanics) because it contains the second derivatives of the $g^{\mu\nu}$, and these cannot be eliminated from the a^l by means of the fundamental equations."[33] From here, Noether makes use of the identity $Div(e^l - i^l) = 0$ to count the number of equations that the components of e^l need to satisfy, arriving at 120 such conditions. She then shows that the identical argument goes through for the general Lagrangian H, so that Hilbert's energy vector can always be decomposed as above.

Noether next takes up a similar analysis of Einstein's version of the energy laws in general relativity, published in [Einstein 1916a], arriving at very similar results. She begins by noting how Einstein bases his theory on the demand that the equations of motion be given by (5.1). Noether then rewrites Einstein's matter-free field equations (5.4) with only two small notational differences: her spacetime coordinates appear as w_α instead of x_α, and she suppresses the coefficient -2χ in the second equation, which Einstein introduced for physical reasons. Likewise, she writes Einstein's law for conservation of momentum and energy (5.7) in the form

$$\sum_l \frac{\partial(t_s^l + T_s^l)}{\partial w_l} = 0.$$

Drawing on Hilbert's notation, and noting that for $\sqrt{-g} = 1,\ H = K$, she writes for the Lagrangian derivative in (5.4):

$$-[\sqrt{g}H]_{\mu\nu} = \sum_\alpha \frac{\partial}{\partial w_\alpha} \left\{ \frac{\partial H}{\partial g_\alpha^{\mu\nu}} \right\} - \frac{\partial H}{\partial g^{\mu\nu}}. \tag{6.10}$$

Noether now connects (6.10) with Einstein's pseudotensor for gravitational energy t_σ^α in (5.5). Multiplying (6.10) by $g_\sigma^{\mu\nu}$ and summing over the indices μ, ν yields:

$$-\sum_{\mu\nu} g_\sigma^{\mu\nu} [\sqrt{g}H]_{\mu\nu} = \frac{\partial}{\partial w_\alpha} \sum_{\mu\nu} g_\sigma^{\mu\nu} \frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \frac{\partial H}{\partial w_\sigma},$$

and thus

$$-\sum_{\mu\nu} g_\sigma^{\mu\nu} [\sqrt{g}H]_{\mu\nu} = \sum_\alpha \frac{\partial t_\sigma^\alpha}{\partial w_\alpha} \tag{6.11}$$

in view of (5.5).

Using Hilbert's Theorem III and the identity (6.4), Noether next obtains:

$$\sum_{\mu\nu} [\sqrt{g}H]_{\mu\nu} p^{\mu\nu} = \sum_{sl} \frac{\partial i_s^l}{\partial w_l} p^s + \sum_{sl} i_s^l p_l^s, \tag{6.12}$$

and in place of (6.3) she writes:

$$-2 \sum_\mu [\sqrt{g}H]_{\mu s} g^{\mu l} = t_s^l + r_s^l. \tag{6.13}$$

[33] Hilbert introduced a^l in a purely formal manner; see the discussion of equation (4.10) above.

Her claim is that $i^l_s = t^l_s + r^l_s$ and that $Div(i^l_s) = Div(t^l_s)$, so that $Div(r^l_s) \equiv 0$. She proves this by multiplying (6.11) by p^s and (6.13) by p^s_l, and then adding these two equations to get:

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}H]_{\mu\nu} p^{\mu\nu} = \sum_l \frac{\partial t^l_s}{\partial w_l} p^s + \sum_{sl} (t^l_s + r^l_s) p^s_l. \tag{6.14}$$

Comparing coefficients in (6.12) and (6.14), Noether deduces the equations:

$$\sum_l \frac{\partial i^l_s}{\partial w_l} = \sum_l \frac{\partial t^l_s}{\partial w_l};\ i^l_s = t^l_s + r^l_s, \tag{6.15}$$

from which follows that

$$\sum_l \frac{\partial r^l_s}{\partial w_l} = Div(r^l_s) \equiv 0,$$

under the assumption that $\sqrt{-g} = 1$ holds.

Summarizing, she concludes that the Einsteinian gravitational pseudotensor t^l_s also decomposes into two parts, $t^l_s = i^l_s - r^l_s$, where by (6.3) i^l_s vanishes as a consequence of the field equations, whereas the divergence of r^l_s vanishes identically, i.e., independent of the field equations. Noether actually shows that $i^l_s = t^l_s + r^l_s$, the second equation in (6.15), is equivalent to Einstein's field equations written in the form (5.8). Finally, she briefly notes that the same considerations hold in the presence of matter, just as in the case of Hilbert's theory.

Humm's copy of Noether's manuscript contains no date, so we can only fix bounds for the period during which she must have written it. In his correspondence with Einstein from late May and early June of 1916, Hilbert alluded to Noether's efforts to reconcile their approaches to energy laws in general relativity. Much later, in January 1918, Hilbert and Klein both made reference to the results she had obtained more than one year earlier, so probably by December 1916 at the latest. Her text, on the other hand, contains no mention of [Einstein 1916b], which surely circulated in Göttingen soon after its publication in early November 1916. Had she known of this text at the time, Noether would have most likely referred to the arguments Einstein set forth therein. These circumstances suggest that she probably completed her manuscript between June and October of 1916. After this date, Einstein published several times on energy conservation,[34] which proved to be one of the most hotly debated issues in his theory of gravitation. For the Göttingen reception of general relativity, however, the most important of these notes was [Einstein 1916b], to which we now turn.

7. Einstein and Weyl respond to Hilbert

Einstein recognized the importance of deriving his field equations for general relativity from an appropriate variational principle, but he strongly opposed

[34]In addition to [Einstein 1916b], see [Einstein 1918a] and [Einstein 1918b].

Hilbert's effort to link the new theory of gravitation with Mie's electromagnetic theory of matter. He originally thought about addressing this issue in [Einstein 1916a], which quickly came to be regarded as a canonical text for the theory [Gutfreund/Renn 2015]. Among Einstein's posthumous papers, one finds an unpublished appendix written for [Einstein 1916a], in which Einstein adopts Hilbert's variational methods, but with a general matter tensor rather than Hilbert's L. In a footnote, he criticizes Hilbert for adopting Mie's matter function, which was based, of course, on the electrodynamic variables alone [Einstein 1996, 346]. Quite possibly, Einstein withdrew this part of the text so as to avoid any potential polemics. He may have also considered this issue too important to merely appear in an appendix, and so he decided instead to publish a separate note on this topic. In late October 1916 he submitted [Einstein 1916b] for publication in the *Sitzungsberichte der Preußischen Akademie*. This provides a much fuller account of methods for deriving the fundamental equations of general relativity using variational principles. In the introduction, he wrote:

> The general theory of relativity has recently been given in a particularly clear form by H.A. Lorentz and D. Hilbert, who have deduced its equations from one single principle of variation. The same thing will be done in the present paper. But my purpose here is to present the fundamental connections in as perspicuous a manner as possible, and in as general terms as is permissible from the point of view of the general theory of relativity. In particular we shall make as few specializing assumptions as possible, in marked contrast to Hilbert's treatment of the subject ([Einstein 1916b, 165]).

Einstein thus employed Lagrangian equations of the type Hilbert derived earlier, but he began with only some general assumptions about such functions $\mathfrak{H}$ of the field variables $g^{\mu\nu}$, q_ρ and their derivatives. He first noted that the second derivatives $g^{\mu\nu}_{\sigma\tau}$ in $\mathfrak{H}$ could be removed by partial integration, leading to a new Lagrangian $\mathfrak{H}^*$ which satisfies

$$\int \mathfrak{H} d\tau = \int \mathfrak{H}^* d\tau + F,$$

where F is a surface term that can be neglected when the integral is suitably varied. In this way, Einstein was able to substitute $\mathfrak{H}^*$ for $\mathfrak{H}$ in his variational principle

$$\delta\left\{\int \mathfrak{H}\, d\tau\right\} = \delta\left\{\int \mathfrak{H}^*\, d\tau\right\} = 0. \tag{7.1}$$

This leads to the Lagrangian equations:

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}} = 0. \tag{7.2}$$

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial q_{\rho\alpha}}\right) - \frac{\partial \mathfrak{H}^*}{\partial q_\rho} = 0. \tag{7.3}$$

Einstein next assumed $\mathfrak{H}$ can be written $\mathfrak{H} = \mathfrak{G} + \mathfrak{M}$ in order to assert the separate existence of the gravitational field from matter. Furthermore, he assumed that $\mathfrak{M}$ was a function of the four electrodynamic variables q_ρ, their derivatives $q_{\rho\alpha}$ and $g^{\mu\nu}$. His $\mathfrak{G}$ took the form $\mathfrak{G}(g^{\mu\nu}, g^{\mu\nu}_\sigma, g^{\mu\nu}_{\sigma\tau})$, where the coefficients of the $g^{\mu\nu}_{\sigma\tau}$ were linear in the $g^{\mu\nu}$. By introducing a function $\mathfrak{G}^*$ analogous to $\mathfrak{H}^*$, Einstein was able to deduce general gravitational field equations of the form

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}} = \frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}}. \tag{7.4}$$

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{M}}{\partial q_{\rho\alpha}}\right) - \frac{\partial \mathfrak{M}}{\partial q_\rho} = 0. \tag{7.5}$$

Einstein next proceeded to specify further assumptions of his theory. This required that

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu, \quad H = \frac{\mathfrak{H}}{\sqrt{-g}}, \quad G = \frac{\mathfrak{G}}{\sqrt{-g}}, \quad M = \frac{\mathfrak{M}}{\sqrt{-g}}$$

all be invariants under general coordinate transformations. This placed only limited restrictions on the matter fields, but G, up to a constant factor, had to be the Riemann curvature scalar, which entails that $\mathfrak{G}^*$ must also be uniquely determined.[35] In a footnote, Einstein gave an explicit formula for $\mathfrak{G}^*$:

$$\mathfrak{G}^* = \sqrt{-g} \sum_{\alpha\beta\mu\nu} g^{\mu\nu} (\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}). \tag{7.6}$$

This generalizes the Lagrangian (5.3) that Einstein used in [Einstein 1916a]. In his letter to Weyl, cited above, Einstein repudiated (5.3), noting that $\mathfrak{G}^*$ is the required gravitational Lagrangian for generally covariant field equations, as Hilbert had shown.

Einstein then went on to carry out the variation $\int \mathfrak{G}^* \, d\tau$, followed by the usual partial integrations, from which he deduced four identities ($\sigma = 1, 2, 3, 4$):

$$\sum_{\nu\alpha} \frac{\partial^2}{\partial x_\nu \partial x_\alpha}\left(\sum_\mu g^{\mu\nu} \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}\right) \equiv 0. \tag{7.7}$$

From the general field equations (7.4) he then derived

$$\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\sum_\mu g^{\mu\nu} \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}\right) = -(\mathfrak{T}^\nu_\sigma + \mathfrak{t}^\nu_\sigma), \tag{7.8}$$

[35] Einstein never cited a mathematical source for this and other related claims, though he was well aware that his whole theory depended on this uniqueness property (see [Einstein 1916b, 167, footnote 1]). Quite possibly this was part of "folklore" knowledge among experts on the Ricci calculus, in which case Einstein might have picked this up from Marcel Grossmann. In 1920 Hermann Weyl published a proof in an appendix to the fourth edition of [Weyl 1918b] (see [Weyl 1952]), noting that the first proof was given by Hermann Vermeil in 1917; see also [Pauli 1958, 43].

where the terms on the right side of the equations denote

$$\mathfrak{T}_\sigma^\nu = -\sum_\mu \frac{\partial \mathfrak{M}}{\partial g^{\mu\sigma}} g^{\mu\nu}; \ \mathfrak{t}_\sigma^\nu = \frac{1}{2}\left(\mathfrak{G}^* \delta_\sigma^\nu - \sum_{\mu\alpha} \frac{\partial \mathfrak{G}^*}{\partial g_\nu^{\mu\alpha}} g_\sigma^{\mu\alpha}\right).$$

From (7.8) and the identities (7.7), Einstein could now deduce his version of the conservation laws:

$$\sum_\nu \frac{\partial}{\partial x_\nu}(\mathfrak{T}_\sigma^\nu + \mathfrak{t}_\sigma^\nu) = 0. \tag{7.9}$$

As before, he designated the $\mathfrak{T}_\sigma^\nu$ as the energy components of matter, whereas the $\mathfrak{t}_\sigma^\nu$ he regarded as the components of the gravitational energy. In closing, he derived the four equations for the energy components of matter

$$\sum_{\mu\nu} \frac{\partial \mathfrak{T}_\sigma^\nu}{\partial x_\nu} + \frac{1}{2} g_\sigma^{\mu\nu} \mathfrak{T}_{\mu\nu} = 0. \tag{7.10}$$

Einstein emphasized that in deriving the conservation laws (7.9) and (7.10) he needed only the gravitational field equations (7.4) but not the field equations for matter (7.5).

Readers familiar with [Hilbert 1915] surely recognized Einstein's desire to place his variational approach to the fundamental equations of his gravitational theory in the sharpest possible contrast with Hilbert's. He had struggled during the late spring of 1916 to understand how Hilbert constructed his invariant energy vector, but openly admitted that its physical significance eluded him entirely. Apparently he felt no differently one year later when he spoke about it with Rudolf Humm. He wondered how energy could be a vector, but also what sense it made when its very definition was multi-valued, since it contained an arbitrary vector [Rowe 2019, 70].

Einstein also had deep misgivings about Hilbert's methodological approach. Writing to Hermann Weyl shortly after [Einstein 1916b] was published, he confessed:

> To me Hilbert's *Ansatz* about matter appears to be childish, just like an infant who is unaware of the pitfalls of the real world.... In any case, one cannot accept the mixture of well-founded considerations arising from the postulate of general relativity and unfounded, risky hypotheses about the structure of the electron.... I am the first to admit that the discovery of the proper hypothesis, or the Hamilton function, of the structure of the electron is one of the most important tasks of the current theory. The "axiomatic method", however, can be of little use in this. (Einstein to Weyl, 23 November 1916, [Einstein 1996, 366].)

Einstein's letter was written in response to a draft of [Weyl 1917], which employed variational methods to deduce conservation laws in general relativity. Weyl shared Einstein's criticism of Hilbert's theory, especially its reliance on Mie's theory and the assumption of the special matter tensor $T_{\mu\nu} = \frac{\partial L}{\partial g^{\mu\nu}}$. He thus emphasized the provisional nature of all efforts to base gravitational theory on variational principles owing to lack of knowledge about elementary particles. "Under these circumstances," he wrote, "it appears to me important to formulate *a Hamiltonian principle that carries as far as our present knowledge of matter reaches* ..." ([Weyl 1917, 118]).

Weyl's theory combined a general matter function to the gravitational and electromagnetic fields. The field effects are then measured by the action integrals:

$$\int H\,d\omega, \quad \int L\,d\omega,$$

where H is given by (7.6) and

$$L = \frac{1}{2}F_{ik}F^{ik} = \frac{1}{2}g^{ij}g^{kh}F_{ik}F_{jh}, \quad F_{ik} = \frac{\partial\phi_k}{\partial x_i} - \frac{\partial\phi_i}{\partial x_k}.$$

Alongside these field actions, Weyl introduces analogous substance actions given by integrals based on density functions for matter dm and electricity de:

$$\int\left\{dm\int\sqrt{g_{ik}\,dx_i dx_k}\right\}, \quad \int\left\{de\int\phi_i\,dx_i\right\}.$$

All of these ingredients enter into Weyl's "world function" F defined on a given region Ω, for which he postulates that under variations of the field variables that vanish at the boundary of Ω and infinitesimal spacetime displacements of the substance elements this F will be an extremum.

From this postulate, he immediately derives corresponding results for gravitation, electromagnetism, and mechanics. Thus, by varying the g^{ij} while holding the ϕ_i and the worldlines of substance fixed, one gets Einstein's gravitational equations (2.1). Varying the ϕ_i yields the Maxwell-Lorentz equations

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}F^{ik})}{\partial x_k} = J^i = \epsilon\frac{dx_i}{ds}.$$

Finally, varying the worldlines of the substance elements leads to the equations of motion for mass points when acted on by electromagnetic forces

$$\rho\left(\frac{d^2x_i}{ds^2} - \Gamma^i_{hk}\frac{dx_h}{ds}\frac{dx_k}{ds}\right) = p^i. \tag{7.11}$$

Here the p^i are the contravariant components of the force corresponding to the covariant

$$p_i = \sum_k F_{ik}J^k.$$

Weyl remarks further that (7.11) can be shown to follow directly from the other two systems of field equations. He regarded these findings as purely phenomenological deductions analogous to those of classical Hamiltonian mechanics. This approach thus stressed flexibility, and the following year he elaborated on some of these ideas in the first edition of *Raum–Zeit–Materie* [Weyl 1918b].

In section 2 of [Weyl 1917], he introduces a general action integral defined on a region of spacetime Ω for which

$$\int_\Omega (H - M)\, d\omega$$

is an extremum. Here the matter-density action M is closely related to Einstein's energy-momentum tensor T_{ik}; the latter is defined, however, in connection with the total derivative of the former. Weyl's objective is to deduce Einstein's energy-momentum equations for matter (7.10) by an appropriate variation applied to $\mathfrak{M} = M\sqrt{g}$. He was apparently the first author to emphasize that the conservation of energy-momentum in general relativity should be deduced from a variational principle under which the variation of the field quantities is induced by coordinate transformations. In Weyl's language, the field variables are "mitgenommen" by means of infinitesimal coordinate transformations [Weyl 1917, 117]. One year later, Klein and also Noether alluded to earlier work of Sophus Lie, who introduced this method in his new group-theoretic approach to differential equations (see [Hawkins 1991]). Within the context of the calculus of variations, this technique came to be known as Lie variation. As was pointed out by Janssen and Renn, Einstein only gradually came to appreciate the importance of this mathematical technique for field physics ([Janssen/Renn 2007, 863]).

Adopting Weyl's notation, one considers transformations

$$x_i \to x_i + \epsilon\xi_i(x_1, x_2, x_3, x_4)$$

for infinitesimal ϵ and ξ_i, which along with their derivatives vanish on the boundary of integration, and then calculates δg^{ik}, the induced variation of the field quantities:

$$\delta g^{ik} = \epsilon(g^{\alpha k}\frac{\partial \xi_i}{\partial x_\alpha} + g^{i\beta}\frac{\partial \xi_k}{\partial x_\beta}).$$

Weyl then distinguished this δ-variation from a second Δ-variation given by

$$\Delta g^{ik} = \delta g^{ik} - \epsilon\frac{\partial g^{ik}}{\partial x_\alpha}\xi_\alpha. \tag{7.12}$$

Under a Δ-variation, the domain of definition Ξ for the coordinates (x_1, x_2, x_3, x_4) corresponding to the region Ω remains identical, leading to what Weyl calls a virtual displacement.

Using this variational technique, Weyl rederives Einstein's energy-momentum equations for matter (7.10) ([Weyl 1917, 124]). Writing dx for $dx_1 dx_2 dx_3 dx_4$, he notes that $\int \mathfrak{M}\, dx$ is an invariant and that

$$\int_\Xi \Delta\mathfrak{M}\, dx = 0.$$

Furthermore,

$$\int_\Xi \Delta\mathfrak{M}\, dx = \int \mathfrak{T}_{ik}\, \Delta g^{ik}\, dx, \quad \mathfrak{T}_{ik} = \sqrt{g}T_{ik}.$$

Substituting (7.12) and carrying out the partial integration leads to

$$\int\Big\{\sum_{krs}\frac{\partial\mathfrak{T}_i^k}{\partial x_k}+\frac{1}{2}\frac{\partial g^{rs}}{\partial x_i}\mathfrak{T}_{rs}\Big\}\xi_i dx=0,$$

and since ξ_i is arbitrary, he gets (7.10):

$$\sum_{krs}\frac{\partial\mathfrak{T}_i^k}{\partial x_k}+\frac{1}{2}\frac{\partial g^{rs}}{\partial x_i}\mathfrak{T}_{rs}=0.$$

Weyl next points out that a parallel argument using the gravitational action H leads to four analogous equations satisfied by the Einstein tensor $R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$. Written in modern notation, these are

$$(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)_{;\nu}=0,$$

known today as contracted Bianchi identities. Since these are formally identical to the equations (7.10), Weyl made the noteworthy observation that the latter equations are an immediate consequence of Einstein's field equations (2.1), written in the form

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=-\kappa T^{\mu\nu}. \tag{7.13}$$

He further observed that this was a natural consequence of a generally covariant theory, since the freedom to choose any coordinate system is reflected in the fact that these ten gravitational field equations satisfy four differential identities. Although Weyl clearly recognized the connection between his results and Hilbert's Theorem I, he made no direct comments about the latter. Instead, he cited Hilbert's second note [Hilbert 1917], which addressed the problem of causality in GR while proposing a method for handling Cauchy problems. Nor did he draw any clear distinction between relativistic conservation laws and their counterparts in classical mechanics. Working within this novel context, Weyl's focus was on adapting variational principles to the new field physics, following the lead of Hilbert, Lorentz, and Einstein. None of these mathematicians and physicists was deeply versed in the fine points of Ricci's tensor calculus, including the full Bianchi identities.[36] Due to this circumstance, they came to regard the contracted Bianchi identities as a result obtained by using variational methods.

8. Klein's Critique of [Hilbert 1915]

The papers by Einstein and Weyl discussed above were carefully studied by Felix Klein, who from early 1917 began to play an active role in ongoing discussions of conceptual problems in general relativity. As noted above in section 3, in January 1918 Klein and Hilbert reached a first consensus regarding some fundamental issues related to general relativistic physics [Klein 1918a].

[36] As was pointed out in [Pais 1982, 274–276]; for the ensuing history, see [Rowe 2018, 263–272].

With regard to variational methods and conservation laws derived from them, Klein emphasized the importance of separating formal deductions from physical claims, such as those that form the basis for Einstein's new gravitational theory. Much of what he and Hilbert discussed centered on the distinction between theories based on invariants of the orthogonal group and those that arise from a variational problem based on general invariants, as in Hilbert's adaptation of Einstein's theory.

Klein introduced a special Lagrangian in place of L, namely

$$L = \alpha Q = -\alpha \sum_{\mu\nu\rho\sigma} (q_{\mu\nu} - q_{\nu\mu})(q_{\rho\sigma} - q_{\sigma\rho})(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),$$

where $-\alpha$ is Einstein's $\kappa = \frac{8\pi K}{c^2}$ and K the universal gravitational constant from Newton's theory [Einstein 1916a, 333]. He then observes that the tiny value $-\alpha = 1.87 \cdot 10^{-27}$ will ensure that the new theory accords with Maxwell's theory, for which $\alpha = 0$. Klein next takes the two integrals separately:

$$I_1 = \int K d\omega, \; I_2 = \alpha \int Q d\omega,$$

and carries out the variation in a purely formal manner, writing:

$$\delta I_1 = \int K_{\mu\nu} \delta g^{\mu\nu} d\omega;$$

$$\delta I_2 = \alpha \int (\sum_{\mu\nu} Q_{\mu\nu} \delta g^{\mu\nu} + \sum_{\rho} Q^{\rho} \delta q_{\rho}) d\omega.$$

Here $K_{\mu\nu}$ is Hilbert's $[\sqrt{g}K]_{\mu\nu} : \sqrt{g}$, whereas $Q_{\mu\nu} = (\frac{\partial \sqrt{g} Q}{\partial g^{\mu\nu}}) : \sqrt{g}$, and the vector

$$Q^{\rho} = -\sum_{\sigma} \frac{\partial(\frac{\partial\sqrt{g}Q}{\partial q^{\rho\sigma}})}{\partial w^{\sigma}} : \sqrt{g}.$$

Clearly the $Q_{\mu\nu}$ are the coefficient's in (4.5), Hilbert's expression for electromagnetic energy, so Klein called these the energy components of the electromagnetic field. He further identified $Q^{\rho} = 0$ as the counterpart to the Maxwell equations.

Carrying out the variation for I_1 leads almost immediately to the four differential equations that Hilbert had derived using Theorem III (see (6.5)):

$$\sqrt{g} \sum_{\mu\nu} K_{\mu\nu} g^{\mu\nu}_{\sigma} + 2 \sum_{\mu\nu} \frac{\partial(\sqrt{g} K_{\mu\sigma} g^{\mu\nu})}{\partial w^{\nu}} = 0, \; \sigma = 1, 2, 3, 4, \tag{8.1}$$

which Klein summarizes in the statement that the vectorial divergence of $K_{\mu\nu}$ vanishes. For the variation of I_2 he obtains:

$$\sum_{\mu\nu} (\sqrt{g} Q_{\mu\nu} g^{\mu\nu}_{\sigma} + 2 \sum_{\mu\nu} \frac{\partial \sqrt{g}(Q_{\mu\sigma} g^{\mu\nu})}{\partial w^{\nu}}) + \sum_{\rho} (\sqrt{g} Q^{\rho} (q_{\rho\sigma} - q_{\sigma\rho})) = 0, \; \sigma = 1, 2, 3, 4. \tag{8.2}$$

Only at this point does Klein make use of the field equations, which here appear in the form:

$$K_{\mu\nu} + \alpha Q_{\mu\nu} = 0;\ \ Q^\rho = 0.$$

Multiplying (8.2) by α and adding this to (8.1) yields:

$$\sum_{\mu\nu} \sqrt{g}(K_{\mu\nu}+\alpha Q_{\mu\nu})g^{\mu\nu}_\sigma + 2\sum_{\mu\nu} \frac{\partial(\sqrt{g}(K_{\mu\sigma}+\alpha Q_{\mu\nu})g^{\mu\nu})}{\partial w^\nu} + \alpha \sum_\rho (\sqrt{g}Q^\rho(q_{\rho\sigma} - q_{\sigma\rho})) = 0. \tag{8.3}$$

From the equations (8.3) Klein immediately deduces that the four equations $Q^\rho = 0$ follow directly from the ten equations $K_{\mu\nu} + \alpha Q_{\mu\nu} = 0$. If, on the other hand, one takes the generalized Maxwell equations $Q^\rho = 0$ alongside the four identities (8.2), then one can conclude that the energy components $Q_{\mu\nu}$ have a vanishing vectorial divergence.

This straightforward analysis pointed to one of the glaring weaknesses in Hilbert's theory, namely the use he made of Theorem 1 to deduce four identities from his fourteen fundamental equations. Hilbert's idea of reducing electrodynamics to gravitational effects hinged on applying Theorem 1 to the world function $H = K + L$. What Klein simply pointed out was that by handling gravity and electromagnetism separately, one can derive four identities from each, namely the four Lagrangian equations derivable from $\delta \int K d\omega = 0$ and $\delta \int Q d\omega = 0$, respectively. This meant that Hilbert's Theorem I led to *eight* identities and not just four, an observation that seriously undermined his unification program.

Klein was also able to shed new light on Hilbert's invariant energy vector e^ν by slightly transforming equations (8.3). This led to the recognition that e^ν could be decomposed into a sum of two vectors, the first being

$$e_1^\nu = -2\sum_{\mu\sigma}((K_{\mu\sigma} + \alpha Q_{\mu\sigma})g^{\mu\nu} + \frac{\alpha}{2}Q^\nu q_\sigma)p^\sigma$$

and the second e_2^ν having vanishing divergence.[37] Since the first vector vanishes by virtue of the field equations, Klein concludes that Hilbert's invariant energy theorem (4.7) is merely an identity and thus by no means analogous to conservation of energy in classical mechanics. These findings were clearly in accord with what Emmy Noether had already pointed out to Hilbert more than a year before. Since she still had her manuscript, she was able to show her derivation to Klein. This probably took place on or shortly after 22 January 1918, when he spoke about these matters at a meeting of the Göttingen Mathematical Society. After mentioning her earlier results, Klein somewhat dismissively wrote that she had not brought out their importance as decisively as he had done in his lecture [Klein 1918a, 559].

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[37] Klein also noted certain properties of e_2^ν, but he found it too difficult to calculate directly. A few months later he discovered a different way to derive Hilbert's e^l and presented this in [Klein 1918b].

9. Klein's Correspondence with Einstein on Energy Conservation

Klein's open letter to Hilbert contained similar remarks about Einstein's derivation of the "conservation laws" (7.9) in [Einstein 1916b] (the quotation marks are Klein's). Klein claimed that these four equations should also be regarded as mathematical identities, by which he apparently meant that they were consequences of the field equations. This assertion was disputed by Einstein and led to some lengthy exchanges between him and Klein during the month of March 1918. On 13 March, Einstein wrote him:

> It was with great pleasure that I read your extremely clear and elegant explanations regarding Hilberts first note. However, I consider your remark about my formulation of the conservation laws to be inaccurate. For equation [(7.9)] is by no means an identity, any more than [(7.8)]; only [(7.7)] is an identity. The conditions [(7.8)] are the mixed form of the field equations of gravitation. [(7.9)] follows from [(7.8)] on the basis of the identity [(7.7)]. The relations here are exactly analogous to those of nonrelativistic theories. [Einstein 1998b, 673]

Einstein might have noticed that what Klein meant by an identity differed from his own understanding, but he was mainly intent on spelling out the physical importance of the pseudo-tensor $\mathfrak{t}_\sigma^\nu$ in the conservation laws (7.9) The $\mathfrak{t}_\sigma^\nu$'s not only lead to these laws but also with (7.8) they provide a physical interpretation entirely analogous to Gauss's law in electrostatics.

> In the static case the number of "lines of force" running from a physical system to infinity is, according to [(7.8)], only dependent on the 3-dimensional spatial integrals
>
> $$\int (\mathfrak{T}_\sigma^\nu + \mathfrak{t}_\sigma^\nu) dV$$
>
> to be taken over the system and the gravitational field belonging to the system. This state of affairs can be expressed in the following way. As far as its gravitational influence at a great distance is concerned, any (quasi-static) system can be replaced by a point mass. The gravitational mass of this point mass is given by
>
> $$\int (\mathfrak{T}_4^4 + \mathfrak{t}_4^4) dV$$
>
> i.e., by the total energy (more precisely, total "rest energy") of the system, exactly as the inertial mass of the system. ...
>
> From [(7.9)] it can be concluded that the same integral $\int (\mathfrak{T}_4^4 + \mathfrak{t}_4^4) dV$ also determines the system's inertial mass. Without the introduction and interpretation of $\mathfrak{t}_\sigma^\nu$, one cannot see that the inertial and gravitational mass of a system agree.
>
> I hope that this anything but complete explanation will enable you to guess what I mean. Above all, though, I hope you will

abandon your view that I had formulated an identity, that is, an equation that places no conditions on the quantities in it, as the energy law. [Einstein 1998b, 674]

Regarding this last point, Klein was still thoroughly unpersuaded, and so he sent Einstein his "rebuttal" in a long letter from 20 March [Einstein 1998b, 685–688]. Klein's key assertion was that the equations (7.9) are completely equivalent to

$$\sum_{\nu} \frac{\partial(K^{\nu}_{\sigma} + \alpha Q^{\nu}_{\sigma})}{\partial w^{\nu}} = 0$$

and that the latter are "physically contentless." He meant by this nothing more than the observation that Einstein's conservation laws followed directly from the gravitational field equations.

Klein further informed Einstein that Carl Runge had found a way to particularize the coordinate system to obtain conserved quantities directly from:

$$\sum_{\nu} \frac{\partial T^{\nu}_{\sigma}}{\partial x_{\nu}} = 0.$$

Delighted by this apparent breakthrough ("the pure egg of Columbus"), he was anxious to learn what Einstein thought about Runge's finding. Emmy Noether already knew about this proposal, and she was highly skeptical. She was visiting her father in Erlangen, so Klein mailed her a draft of [Klein 1918a].along with a description of Runge's result. She quickly went to work and found from concrete examples that Runge's coordinate transformation led to well-known identities that cannot be interpreted as energy laws.[38]

Einstein clarified his views on these matters in a letter from 24 March. In this reply, he underscored that the equations above contained *part* of the content of the field equations

$$K^{\nu}_{\sigma} + \alpha Q^{\nu}_{\sigma} = 0.$$

The same was true for the equations

$$\sum_{\nu} \frac{\partial(\mathfrak{T}^{\nu}_{\sigma} + \mathfrak{t}^{\nu}_{\sigma})}{\partial x_{\nu}} = 0,$$

though with the important advantage that these equations can be used to obtain an integral formulation for energy conservation on regions of space-time over which the $\mathfrak{T}$'s and $\mathfrak{t}$'s vanish. One then obtains

$$\frac{d}{dx_4}\{\int (\mathfrak{T}^{4}_{\sigma} + \mathfrak{t}^{4}_{\sigma})\} = 0.$$

Einstein emphasized that "the temporal constancy of these four integrals is a nontrivial consequence of the field equations and can be looked upon as entirely similar and equivalent to the momentum and energy conservation laws in the classical mechanics of continua" [Einstein 1998b, 697].

[38]E. Noether to F. Klein, 12 March 1918, Nachlass Klein, (SUB), Göttingen.

As for Runge's proposal for obtaining the conservation laws by particularizing the coordinate system, Einstein reported that he had explored that idea himself, but had given it up "because the theory predicts energy losses due to gravitational waves" and these loses could not be taken into account. Einstein included an offprint of his recent paper [Einstein 1918a], in which he introduced the quadrupole formula for the propagation of gravitational radiation. This was a typical instance showing how Einstein could quickly cast aside a mathematical idea when he noticed that it failed to conform to his physical understanding. Emmy Noether's reservations regarding Runge's approach were, of course, based on essentially mathematical considerations. Klein and Runge soon hereafter dropped this line of investigation, but Klein continued to explore the mathematical underpinnings of energy conservation in the context of invariant variational principles.

In mid-July, he wrote to Einstein with news of a first breakthrough: "I have succeeded in finding the organic law of construction for Hilbert's energy vector" [Einstein 1998b, 833]. Klein's innovation was surprisingly simple. Previously, he and others has carried out infinitesimal variations using a vector field p^τ, which along with its derivatives was required to vanish on the boundary of the integration domain. Klein now dropped this restriction, so that in carrying out the variation he obtained an additional triple integral of the form

$$\int\int\int \sqrt{g}\{e^1 dw^2 dw^3 dw^4 + \cdots + e^4 dw^1 dw^2 dw^3\}.$$

He then found that Hilbert's energy vector was essentially identical to (e^1, e^2, e^3, e^4), differing only by terms with vanishing divergence. In his letter to Einstein, Klein reported that he hoped now to find his way to Einstein's formulation of energy conservation based on $\mathfrak{T}^\nu_\sigma + \mathfrak{t}^\nu_\sigma$. Einstein answered: "It is very good that you want to clarify the formal significance of the $\mathfrak{t}^\nu_\sigma$. For I must admit that the derivation of the energy theorem for field and matter together appears unsatisfying from the mathematical standpoint, so that one cannot characterize the $\mathfrak{t}^\nu_\sigma$ formally" [Einstein 1998b, 834]. Einstein was also unhappy about the fact that his pseudotensor was unsymmetric, unlike the matter tensor.[39]

It should be emphasized that Klein was working closely with Emmy Noether during this period, as he acknowledged in [Klein 1918a] and [Klein 1918b]. In fact, the latter paper and [Noether 1918b] should be seen as complementary studies, and in today's world would surely have been co-authored publications. On Monday, 22 July, Klein spoke on "Hilberts Energievektor" before the Göttingen Mathematical Society, one day before Noether's talk on "Invariante Variationsprobleme." Klein then submitted the preliminary version of her findings to the Göttingen Scientific Society on Friday, 26 July, having done the same one week earlier with his manuscript for [Klein 1918b]. Both papers underwent final revision in September and appeared in the *Göttinger Nachrichten* shortly afterward.

[39] In 1951 Landau and Lifschitz introduced a symmetric pseudotensor for gravitational energy; unlike the Einstein pseudotensor, it conserves angular momentum.

By this time, H.A. Lorentz had also derived differential equations for energy conservation in gravitational fields, so his was a third formulation in addition to those of Einstein and Hilbert. It seemed evident that these different versions must be somehow related, and Klein hoped to explain how. Noether's earlier work on the same question clearly helped to move this project forward. Klein's framework in [Klein 1918b] extends the one he utilized in [Klein 1918a]. He now begins with a general variational problem for a scalar function K viewed as a function of $g^{\mu\nu}, g^{\mu\nu}_{\rho}, g^{\mu\nu}_{\rho\sigma}$ alone. In this general setting he derives a series of identities leading to what he calls the principal theorem, which he writes in the form

$$\sum_{\mu\nu} \sqrt{g}(K_{\mu\nu} g^{\mu\nu}_{\tau}) \equiv 2 \sum_{\sigma} \frac{\partial \sqrt{g} U^{\sigma}_{\tau}}{\partial w^{\sigma}}, \tag{9.1}$$

where $K_{\mu\nu}$ is the Lagrangian derivative. This identity effectively turns the four expressions on the left-hand side into what Klein calls elementary divergences because they only involve the first derivative of the $g^{\mu\nu}$. The right-hand side derives from the triple integral above, which Klein introduced in order to derive Hilbert's energy vector. In the previous derivations this expression simply vanishes due to the conditions imposed on the boundary of integration.

Klein gave a simple extension of equation (9.1) after writing it in the abbreviated form:

$$\mathfrak{K}_{\mu\nu} g^{\mu\nu}_{\tau} \equiv 2 \frac{\partial \mathfrak{U}^{\sigma}_{\tau}}{\partial w^{\sigma}}.$$

He then noted that the Lagrangian derivative of any elementary divergence $\mathfrak{Div}$ vanishes. So for any $\mathfrak{K}^* = \mathfrak{K} + \mathfrak{Div}$, the left-hand side will remain the same, and the theorem then reads:

$$\mathfrak{K}_{\mu\nu} g^{\mu\nu}_{\tau} \equiv 2 \frac{\partial \mathfrak{U}^{*\sigma}_{\tau}}{\partial w^{\sigma}}.$$

Only at this point does Klein take up analysis of these expressions as invariants of groups. Those deriving from the left-hand side then correspond to invariants under general coordinate transformations (or, as one would say today, arbitrary diffeomorphisms). The U^{σ}_{τ}, resp. $U^{*\sigma}_{\tau}$, on the other hand, are only invariant under affine transformations. This was also the case with Einstein's pseudo-tensor, but Klein now underscored the key property that Einstein had already noted before, namely that these affine invariants enter into an equation that is valid in all coordinate systems. In the present case, this reads:

$$\frac{\partial (\mathfrak{K}^{\sigma}_{\tau} + \mathfrak{U}^{\sigma}_{\tau})}{\partial w^{\sigma}} \equiv 0, \tag{9.2}$$

or from the extended theorem,

$$\frac{\partial (\mathfrak{K}^{\sigma}_{\tau} + \mathfrak{U}^{*\sigma}_{\tau})}{\partial w^{\sigma}} \equiv 0. \tag{9.3}$$

These are evidently purely mathematical deductions valid for any invariant scalar function K.

Klein next turns to physics, by introducing the field equations in the simplest form suitable for his purposes, writing:

$$\mathfrak{K}_\tau^\sigma - \chi\mathfrak{T}_\tau^\sigma = 0.$$

Substituting in (9.2) and (9.3), leads to two forms of the conservation laws,

$$\frac{\partial(\mathfrak{T}_\tau^\sigma + \frac{1}{\chi}\mathfrak{U}_\tau^\sigma)}{\partial w^\sigma} = 0,$$

$$\frac{\partial(\mathfrak{T}_\tau^\sigma + \frac{1}{\chi}\mathfrak{U}_\tau^{*\sigma})}{\partial w^\sigma} = 0.$$

The first of these reflects the form Lorentz derives, whereas Klein shows that Einstein's formulation (7.9) conforms with the second. Analyzing Hilbert's energy vector led to additional complications, but the net result was the same: except for additional terms of no physical significance, its form was also of the second type.

Soon after Klein's paper [Klein 1918b] on the differential form of the conservation laws came out in October, he sent a copy to Einstein. The latter responded with enthusiasm: "I have already studied your paper most thoroughly and with true amazement. You have clarified this difficult matter fully. Everything is wonderfully transparent" [Einstein 1998b, 917]. He was particularly delighted that Klein had not rejected his controversial pseudotensor for gravitational energy. Only one question still bothered him: how can one prove that Hilbert's expression is truly a generally covariant vector?

Klein answered with a calculation, but Einstein found the argument behind it insufficient [Einstein 1998b, 932]. One week later, after consulting with his *Assistent* Hermann Vermeil, Klein sent Einstein a new calculation. He realized that the argument was anything but elegant, but was eager to learn what Einstein thought of it [Einstein 1998b, 936–937]. He received this immediate response:

> Thank you very much for the transparent proof, which I understood completely. The fact that it cannot be realized without calculation does not detract from your overall investigation, of course, since you make no use of the vector character of e^σ.– In the whole theory, one thing still disturbs me formally, namely, that $T\mu\nu$ must necessarily be symmetric but not $t\mu\nu$, even though both must enter equivalently in the conservation law. Maybe this disparity will disappear when "matter" is included, and not just superficially as it has been up to now, but in a real way in the theory. [Einstein 1998b, 938].[40]

A few days later, Klein had the opportunity to discuss this problem with Emmy Noether, who explained that Hilbert had already alluded to a general method for proving that e^σ transformed as a vector in [Hilbert 1915]. Klein immediately wrote to Einstein with a sketch of the proof, which did

[40] Landau and Lifschitz introduced a symmetric pseudotensor for gravitational energy in 1951.

not depend on special properties of K [Einstein 1998b, 942–943]. Once again, Noether emerged as the real expert when it came to unpacking the mysteries surrounding Hilbert's energy vector.

10. Noether's Two Theorems

This was also the case with Hilbert's Theorem I and its role in the formulation of conservation laws. Klein's main concern in [Klein 1918a] was the status of conservation laws in general relativity. Contrary to Einstein, he distinguished sharply between these new findings and traditional conservation laws in classical mechanics. The latter, he argued, cannot simply be deduced from a variational principle; for example, one cannot derive

$$\frac{d(T+U)}{dt} = 0$$

without invoking specific physical properties or principles, such as Newton's law of motion. Klein attached great significance to this issue in part because he wanted to promote ideas from his "Erlangen Program" [Klein 1872], which he was adapting into a general doctrine applicable to the new physics. Relativity theory, according to Klein, should not be thought of exclusively in terms of two groups – the Lorentz group of special relativity and the group of continuous point transformations of general relativity – but rather should be broadly understood as the invariant theory *relative to some given group* that happens to be relevant to a particular physical theory. This was the mathematical context Klein had in mind when he emphasized the distinction between conservation laws in classical mechanics, special relativity, and the general theory of relativity.[41]

Hilbert not only agreed with Klein's assertion, he went even further by expressing the opinion that the lack of analogy between classical energy conservation and his own energy equation was a characteristic feature of general relativity. In his own inimitable manner, he even claimed *one could prove a theorem* effectively ruling out conservation laws for general transformations analogous to those that hold for the transformations of the orthogonal group. Klein replied by saying: "It would interest me very much to see the mathematical proof carried out that you alluded to in your answer" [Klein 1918a, 565]. Hilbert's conjecture was resolved some months later when Emmy Noether published "Invariante Variationsprobleme" [Noether 1918b].

Noether was in Erlangen around the time Klein was putting the last touches on [Klein 1918a]. From there she wrote him on 29 February 1918: "I thank you very much for sending me your note and today's letter [same day delivery was not uncommon in those times], and I'm very excited about your second note [Klein 1918b]; the notes will certainly contribute much to the

[41] Klein's articles on relativity theory originally appeared in the *Göttinger Nachrichten* as well, but in 1921 he republished them along with additional commentary in the first volume of his collected works. In doing so, he placed them in a special section entitled "Zum Erlanger Programm" ([Klein 1921-23, I: 411–612]).

understanding of the Einstein–Hilbert theory."[42] After this she proceeded to explain where matters stood with regard to the key question Klein hoped to answer, namely the relationship between the classical and relativistic energy equations. Clearly, she was already deeply immersed in this problem.

The fundamental results Noether obtained in [Noether 1918b] not only provided a general proof of Hilbert's Theorem I, they also clarified mathematically how conservation laws arise in Lagrangian systems for classical mechanics as well as modern field theories. In her introduction, she described her approach as one that combined the formal methods of the calculus of variations with techniques from Sophus Lie's theory of continuous groups. Most of Lie's work was motivated by a vision for solving general systems of differential equations that admit a given group of transformations. His pursuit of this program led him to develop what came to be known as the theory of Lie groups.[43] Noether pointed out that within the context of invariant variational systems one could obtain much stronger theorems than in the general cases handled by Lie.

Noether's "theorem" is really two theorems, one dealing with transformation groups determined by finitely many parameters, the other concerned with groups determined by finitely many *functions* and their derivatives. Following Lie, she called the first type a finite continuous group, the second an infinite continuous group. Of particular significance are those groups containing both types of structures, which Lie called mixed groups. With regard to physical interpretations, she noted that her first theorem generalized the formalism underlying the standard results pertaining to first integrals in classical mechanics, whereas her second theorem constituted "the most general group-theoretic generalization of 'general relativity"' [Noether 1918b, 240].

She formulated these two theorems as follows:

> Theorem I. Let G_ρ be a finite continuous group with ρ parameters. If the integral I is invariant with respect to G_ρ, then ρ linearly independent combinations of the Lagrangian expressions become divergences, and conversely. The theorem also holds in the limiting case of infinitely many parameters.
>
> Theorem II. Let $G_{\infty\rho}$ be an infinite continuous group depending on ρ continuous functions. If the integral I is invariant with respect to $G_{\infty\rho}$, in which arbitrary functions and their derivatives up to the σth order appear, then ρ identical relations are satisfied between the Lagrangian expressions and their derivatives up to the order σ. The converse also holds here. [Noether 1918b, 238–239]

Theorem I ("Noether's Theorem") is usually the only result cited in the physics literature. Its importance for physical theories is that it precisely characterizes how conserved quantities arise from symmetries in variational

[42] E. Noether to F. Klein, 29 February 1918, Nachlass Klein, (SUB), Göttingen.

[43] For historical background on Lie's work and its influence, see [Hawkins 1991] and [Hawkins 2000].

systems. In a letter to Einstein from 7 January 1926, Noether wrote that "for me, what mattered in 'Invariante Variationsprobleme' was the precise formulation of the scope of the principle and, above all, its converse ..." [Kosmann-Schwarzbach 2006/2011, 2011: 164]. Likewise, Theorem II characterizes the manner in which identities satisfied by a combination of the Lagrangian expressions and their derivatives come into play. Hilbert's Theorem I may thus be seen as a special case of Noether's second theorem corresponding to transformations of the group $G_{\infty 4}$ given by four functions that depend on the four coordinates of the world–points.

Noether combined these two key results in order to distinguish between "proper" and "improper" conservation laws in physics. Suppose the integral I is invariant with respect to a group $G_{\infty\rho}$. One can then particularize the functions p_λ, $\lambda = 1, 2, \ldots, \rho$ to obtain a *finite* continuous subgroup G_σ of $G_{\infty\rho}$. The divergence relations that arise will then be fully determined by this G_σ. Moreover, the divergence relations associated with G_σ must, being a subgroup of $G_{\infty\rho}$, also be derivable from identities connecting the Lagrangian expressions and their total derivatives by suitably particularizing the p_λ. Noether called such relations that were derivable from an infinite group $G_{\infty\rho}$ improper ("uneigentliche") divergence relations; all others were proper ("eigentliche"). From these considerations, she concluded that:

> The divergence relations corresponding to a finite group G_σ are improper if and only if G_σ is a subgroup of an infinite group with respect to which I is invariant. [Noether 1918b, 254]

As Noether noted, the conservation laws of classical mechanics as well as those of special relativity theory are proper in the above sense. One cannot deduce these as invariants of a suitably particularized subgroup of an infinite group. In general relativity, on the other hand, every Lagrangian variational formalism will lead to four identities as a consequence of the principle of general covariance. In summarizing these findings, Noether wrote:

> Hilbert expressed his assertion regarding the absence of actual energy theorems as a characteristic attribute of 'general relativity theory.' If this assertion is to be literally valid, then the term 'general relativity' must be taken more broadly than is usual and extended to groups that depend on n arbitrary functions. [Noether 1918b, 256–257]

From the mathematical standpoint, Noether's analysis provided a strikingly clear and altogether general answer to the question Klein had raised about the status of conservation laws in Lagrangian systems. Her study [Noether 1918b] remains today nearly the last word on this subject.[44]

[44]For remarks on modern refinements of Noether's results, see [Kosmann-Schwarzbach 2006/2011].

11. On Hilbert's Revised Theory in [Hilbert 1924]

Klein took deep satisfaction in the part he was able to play in elucidating the mathematical underpinnings of key results in the general theory of relativity. In a letter to Pauli, he related Einstein's remark about how Klein's third note on general relativity [Klein 1919] had made him "as happy as a child whose mother had presented him with a piece of chocolate," adding that "Einstein is always so gracious in his personal remarks, in complete contrast to the foolish promotional efforts ("törichten Reklametum") undertaken to honor him."[45] Klein also made it clear to Pauli that his article "could not pass over Hilbert's efforts in silence."[46] Pauli was skeptical of the various unified field theories that had been advanced by Mie, Weyl, and Einstein [Pauli 1958, 205–206]. Regarding Mie's theory, he saw no way to deduce the properties of the world function L just by knowing its invariants; there were simply far too many alternatives [Pauli 1958, 189–190]. By this time, Hilbert had probably drawn a similar conclusion.

Nevertheless, three years later he published a revised version of [Hilbert 1915] and [Hilbert 1917] in *Mathematische Annalen* [Hilbert 1924]. There, he advertised this as "essentially a reprint of the earlier communications ... and my remarks on them that F. Klein published in ... [Klein 1918a] – with only minor editorial alterations and changes in order to ease understanding" [Hilbert 1924, 1]. In truth, however, this "reprint" contains major changes that no careful reader could possibly miss. These pertain mainly to [Hilbert 1915], the focus of discussion for the present account.

As noted above, Hilbert's invariant energy vector disappears entirely in this revised account. Furthermore, he softened the physical claim that had been so central for the original theory, namely "electrodynamic phenomena are the effects of gravitation." In place of this, he now wrote that the four independent identities that derive from the gravitational equations $[\sqrt{g}H]_{\mu\nu} = 0$ signify the "connection between gravity and electrodynamics" [Hilbert 1924, 10]. Hilbert noted that his earlier Theorem I had served as the leitmotiv for his theory, but he only mentions it in passing. In a footnote, he cites Emmy Noether's paper [Noether 1918b] for a "general proof" [Hilbert 1924, 6]. In its place, he refers to a slightly more general version of the result formerly called Theorem III. Here it appears as Theorem 2, but instead of the expressions (6.2) and (6.3) Hilbert now writes:

$$i_s = \sum_{\mu\nu}([\sqrt{g}J]_{\mu\nu}g_s^{\mu\nu} + [\sqrt{g}J]_{\mu}q_{\mu s})\,;\; i_s^l = -2\sum_{\mu}[\sqrt{g}K]_{\mu s}g^{\mu l} + [\sqrt{g}J]_l q_s. \tag{11.1}$$

As before, (6.5) still holds, and he now applies this theorem successively to K and L, following Klein (and implicitly Noether). This leads in the first

[45]Klein to Pauli, 8 March 1921, in [Pauli 1979, 79].
[46]Klein to Pauli, 8 May 1921, in [Pauli 1979, 31].

case to the four identities (8.1):

$$\sum_{\mu\nu}[\sqrt{g}K]_{\mu\nu}g_s^{\mu\nu}+2\sum_{\mu m}\frac{\partial([\sqrt{g}K]_{\mu s}g^{\mu m})}{\partial x_m}=0,\ s=1,\,2,\,3,\,4.$$

Hilbert also rewrites the field equations (4.4) by introducing

$$T_{\mu\nu}=-\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}.$$

Since $[\sqrt{g}K]_{\mu\nu}=\sqrt{g}(K_{\mu\nu}-\frac{1}{2}g_{\mu\nu}K)$, the field equations now appear as:

$$K_{\mu\nu}-\frac{1}{2}g_{\mu\nu}K=T_{\mu\nu}.$$

Inserting L in (6.5) yields

$$\sum_{\mu\nu}-(\sqrt{g}T_{\mu\nu})g_s^{\mu\nu}+2\sum_{m}\frac{\partial(-\sqrt{g}T_s^m)}{\partial x_m}+\sum_{\mu}[\sqrt{g}L]_\mu q_{\mu s}-\sum_{\mu}\frac{\partial([\sqrt{g}L]_\mu q_s)}{\partial x_\mu}=0. \tag{11.2}$$

Invoking the field equations $[\sqrt{g}L]_\mu=0$, the last two terms vanish, leaving:

$$\sum_{\mu\nu}\sqrt{g}T_{\mu\nu}g_s^{\mu\nu}+2\sum_{m}\frac{\partial\sqrt{g}T_s^m}{\partial x_m}=0, \tag{11.3}$$

which are the familiar equations (7.10) for the matter tensor $T_{\mu\nu}$. As Einstein noted in [Einstein 1916a, 325], these equations are the general relativistic analogue for the classical conservation laws of momentum and energy, where the second term represents the transfer of momentum-energy from the gravitational field to matter.[47] Hilbert remarks accordingly that these equations pass over to true conservation laws when the $g^{\mu\nu}$ are constant, in which case

$$\sum_{m}\frac{\partial T_s^m}{\partial x_m}=0.$$

Hilbert showed similarly that by invoking the gravitational field equations (4.4) the first two terms above vanish, which leaves:

$$\sum_{\mu}[\sqrt{g}L]_\mu q_{\mu s}-\sum_{\mu}\frac{\partial([\sqrt{g}L]_\mu q_s)}{\partial x_\mu}=0.$$

These are four differential equations connecting the electrodynamical Lagrangians with their first derivatives, as asserted by Noether's second theorem. Hilbert's derivation at this key point in [Hilbert 1924] follows the argument in [Klein 1918a] almost to the letter. Thus many of the technical tricks he employed in [Hilbert 1915] have now disappeared, making this paper far easier to follow than the original.

It would seem unlikely that many readers noticed that [Hilbert 1924] was hardly what could be called "essentially a reprint of the earlier communications." Yet even later commentators accepted this characterization at

[47] This passage is the only place in [Einstein 1916a] in which Einstein made direct reference to [Hilbert 1915].

face value, as pointed out in [Rowe 1999, 227]. In today's world, with our ready access to so many published sources, one might hope that historians would be held to a higher standard.

At the outset of his paper, Hilbert noted that only future research could decide whether a program like the one he first envisioned in 1915 might actually be realizable. Many physicists would continue to cling to this dream of establishing a pure field theory that could account for microphysical phenomena, but a growing number had become skeptical. By 1924, Hilbert had begun to immerse himself in the foundational problems of quantum theory, and these would occupy a good part of his attention throughout the 1920s [Sauer/Majer 2009, 503–706]. Emmy Noether, on the other hand, would soon emerge to become the leader in Göttingen of an important research school, one whose followers promoted her special vision for abstract algebra. Her venture into mathematical physics, fruitful as it had been, was merely an episode in her early career. If physicists today think of her in connection with "Noether's Theorem" – by which they mean the first and not the highly significant second theorem in [Noether 1918b] – they typically overlook the role she played in the dramatic, but also highly complex story of how Einstein's theory of gravitation was received in Germany during the years of the First World War.

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