Generalized Rastall's gravity and its effects on compact objects

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We present a generalization of Rastall's gravity in which the conservation law of the energy-moment tensor is altered, and as a result, the trace of the energy-moment tensor is taken into account together with the Ricci scalar in the expression for the covariant derivative. Afterwards, we obtain the field equation in this theory and solve it by considering a spherically symmetric space-time. We show that the external solution has two possible classes of solutions with spherical symmetry in the vacuum in generalized Rastall's gravity. The first class of solutions is completely equivalent to the Schwarzschild solution, while the second class of solutions has the same structure as the Schwarzschild–de Sitter solution in general relativity. The generalization, in contrast to constant value $k = 8\pi G$ in general relativity, has a gravitational parameter k that depends on the energy density ρ. As an application, we perform a careful analysis of the effects of the theory on neutron stars using realistic equations of state (EoS) as inputs. Our results show that important differences on the profile of neutron stars are obtained within two representatives EoS.

Keywords: Generalization; Rastall's Gravity; Neutron Stars

I. INTRODUCTION

Although general relativity (GR) has been successfully tested in many aspects, some open problems exist in both cosmology and astrophysics. Since the discovery of the discrepancy between the predicted rotation curves of galaxies and the observed motion [1], and the missing mass of galaxy clusters [2], the dark matter hypothesis remains open. Moreover, the accelerated expansion of the universe observed today suggests the existence of the so-called dark energy. Modified theories of gravity have gained attention because they may offer a way to solve these problems considering that these exotic forms of matter and energy are effects of a generalization of the GR due to a modified gravity.

In this sense, the Rastall's Theory of gravity, which may be obtained through a reinterpretation of the the conservation law on the energy-momentum tensor in curved spaces, couples the geometry to the matter in a modified way. Rastall argued that the usual conservation law on the energy-momentum tensor ${T^{\mu}}_{\nu;\,\mu} = 0$ is tested only in the Minkowski spacetime such that in curved space-time it is possible to generalize this expression to ${T^{\mu}}_{\nu;\,\mu} = A_{\nu}$, where the functions A_{ν} vanish in flat space-time. Indeed, one possible implication of the modified conservation law in Rastall's gravity is to see the condition ${T^{\mu}}_{\nu;\,\mu} \neq 0$ as a consequence of the creation of particles in a cosmological context [3]. In astrophysics, the extra degree of freedom due to the modified expression for the divergence of the energy-momentum tensor has been explored in the study of neutron stars [4, 5], where the authors concluded that substantial modifications for the mass-radius relation are obtained even for very small alterations on the parameter of the Rastall's Theory.

Although the field equation in Rastall's theory is a generalization of the field equation in GR, it is well known that the static spherical symmetry solution in vacuum obtained with Rastall's theory coincides with the vacuum solution in GR [6]. In fact, it has been shown that there are two possible classes of solutions with spherical symmetry in vacuum in the Rastall's gravity. The first class of solutions is completely equivalent to the Schwarzschild solution while the second class of solutions has the same structure of the Schwarzschild–de Sitter solution in the GR [6]. But the effect of Rastall's theory is more evident and interesting in the presence of matter or electric charge, that is, $T_{\mu\nu} \neq 0$. Several works involving charged static spherically symmetric black holes and black hole solutions surrounded by fluid [7–13], cosmological problems [14–18], and other theoretical works [19–22]. have been explored in the Rastall's Theory.

In this work, we intend to study a more general modification of the conservation law not yet explored in the usual Rastall gravitational theory. In his original work, Rastall has already mentioned the possibility of relating the conservation law of the energy-moment tensor to A_{ν} with $A_{\nu} = \lambda {\delta^{\mu}}_{\nu} R_{,\,\mu}$. However, it is possible to choose a function A_{ν} that depends on the scalar R and, additionally, on the trace of the energy-momentum tensor T

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in a more general way. Due to the fact that R is associated to the modification of the conservation law, it is natural to assume that the coupling between R and T could also contribute to breaking the conservation law. We propose to choose a general function A_ν as the combination $A_\nu = (\alpha\delta^\mu{}_\nu R + \beta\delta^\mu{}_\nu RT)_{;\,\mu}$. Indeed, we explore this modification of the conservation law and its effects on the resulting solutions of the modified field equation. Similarly, some categories of modified theories of gravity as the $f(R,T)$ theory have considered a field equation that depends on a function of R and T in which the trace could be induced by exotic imperfect fluids or quantum effects [23].

In order to test the theory we use a well known astrophysical lab, the neutron stars. The death of a massive star in a core-collapse supernova can leave as reminiscent a neutron star or a black hole. A typical neutron star has about 1.4 $M_\odot$, a radius of the order of $11 - 13$ km and produces a strong gravitational field that can be used to test new gravity theories in extreme conditions. Additionally to the theoretical point of view, new experiments and observations like the NICER mission [24–26], the LIGO-Virgo gravitational waves observations from neutron star merges [27, 28] together with their electromagnetic counterpart [29, 30] are making the astrophysical constraints to these objects continuously more restrictive, which makes these compact objects even more suitable to be used in tests of alternative gravity theories.

This paper is organized in the form: In Sec. II we review of the Rastall's theory of gravity and then expand the original work by considering a conservation relation that depends on the trace of energy momentum tensor and on the Ricci scalar. We obtain the Newtonian limit of the field equation and study vacuum solutions with spherical symmetry. Neutron star are considered in Sec. III where we analyze the effects of the modified gravity on neutron stars mass and radius profiles using a soft and stiff realistic EoS. Finally, in Sec. IV we show our results.

II. GENERALIZATION OF RASTALL'S THEORY OF GRAVITY

In order to expand the original work and consider a conservation relation that depends on the trace of energy-momentum tensor, we will briefly review the original theory [31]. Then, we will show how to modify Einstein's field equation such that the non-conservative aspect of generalized theory will be taken into account.

A. Rastall's theory

The left hand side of the usual Einstein field equations satisfies $G^\mu{}_{\nu;\,\mu} = 0$, which may be easily verified by using the Bianchi identities. In fact, this relation is in accordance with the right hand side of the field equation if one considers the conservation law $T^\mu{}_{\nu;\,\mu} = 0$. However, in Rastall's gravity [31] it is argued that this equation, in a general space-time, may be replaced by the modified relation $T^\mu{}_{\nu;\,\mu} = \lambda R_{,\,\nu}$ where λ is a constant. After rewriting the terms of this equation we obtain the relation

$$\left(T^\nu{}_\mu - \lambda\delta^\nu{}_\mu R\right)_{;\,\nu} = 0. \tag{1}$$

In this way, Eq. (1) can be used to generalize the Einstein field equation so that the term in brackets in the this equation is used on the right hand side of the field equation. The result is

$$R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R = k\left(T^\nu{}_\mu - \lambda\delta^\nu{}_\mu R\right), \tag{2}$$

where k is the Rastall coupling constant.

B. Generalized Rastall's theory

We discuss now the generalization of Rastall's theory of gravity. We propose that the general function A_ν related to the divergence of the energy momentum tensor in curved space time is given by $A_\nu = (\alpha\delta^\mu{}_\nu R + \beta\delta^\mu{}_\nu RT)_{;\,\mu}$, *i.e.*, it has the same dependence on R as in the original Rastall work, in addition to a coupling term defined by RT, where T is the trace of the energy momentum tensor $T_{\mu\nu}$. In particular, it is expected that the final form of the field equation in this theory will incorporate the elements of this modification and will be able to reproduce the main features of Rastall's gravity in a particular case. As a test for the theory, we solve the field equation that originates from a metric that can be used to model space-time compact stars, such as neutron stars, and thus analyze the possible effects on the mass versus radius diagrams of these objects. As mentioned, the modification in the energy-momentum conservation law has the following form:

$$T^\mu{}_{\nu;\,\mu} = \alpha R_{,\,\nu} + \beta\left(RT\right)_{,\,\nu}, \tag{3}$$

where α and β are called coupling parameters, which measure the deviation from standard theory of GR and quantify the affinity of the matter field coupled with geometry.

The usual Rastall's gravity can be recovered in the appropriate limit of $\beta \to 0$. The divergence of $T_{\mu\nu}$ given by equation (3) is proportional to the gradients of R in both terms. Therefore, in the flat space-time, when $R = 0$, the usual conservation law is recovered. From equation (3) we implement the following expression:

$$\left(T^\nu{}_\mu - \alpha\delta^\nu{}_\mu R - \beta\delta^\nu{}_\mu RT\right)_{;\,\nu} = 0. \tag{4}$$

In fact, assuming the condition given by the above expression, the modified field equation of generalized Rastall's gravity can be written as

$$R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R = k\left(T^\nu{}_\mu - \alpha\delta^\nu{}_\mu R - \beta\delta^\nu{}_\mu RT\right), \tag{5}$$

where k is the modified gravitational coupling constant in this theory. Taking the trace of the previous equation, we have

$$R = \frac{kT}{4k(\alpha + \beta T) - 1}, \tag{6}$$

which leads to the following expression

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k\tau_{\mu\nu}, \tag{7}$$

where $\tau_{\mu\nu}$ is called effective energy-momentum tensor having the following expression:

$$\tau_{\mu\nu} = T_{\mu\nu} - \frac{g_{\mu\nu}T}{4 - \frac{1}{k(\alpha+\beta T)}}. \tag{8}$$

In the next section, we use the Newtonian limit to obtain in this context the form of the gravitational constant k.

C. Newtonian limit

Next we calculate the Newtonian limit of Einstein's field equations so that we can obtain the value of the k constant in our generalized Rastall's theory. To do this, we compare our field equations in the weak field regime with the Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho. \tag{9}$$

In the Newtonian limit we can replace the metric tensor $g_{\mu\nu}$ by the Minkowski tensor $\eta_{\mu\nu}$ in terms that multiply the curvature so that equation (5) reads:

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = kT_{\mu\nu} - k\eta_{\mu\nu}(\alpha R - \beta RT). \tag{10}$$

In this limit we have $\rho \gg p$ and therefore $|T_{00}| \gg |T_{ij}|$ [32], so that if we look at the (00) component of equation (10) we find the following:

$$R_{00} - \frac{1}{2}(-1)R = k(-\rho) - k(-1)(\alpha R - \beta R(-\rho)). \tag{11}$$

Using the approximation $R \approx \sum_{k=1}^{3} R_{kk} - R_{00}$, we can obtain the relation:

$$R = \frac{2R_{00}}{1 - 6k\alpha + 6k\beta\rho} \tag{12}$$

using this relation in equation (11) and knowing that $R_{00} \approx -\nabla^2\phi$, we will find:

$$k\left(\frac{1 - 6k\alpha + 6k\beta\rho}{1 - 4k\alpha + 4k\beta\rho}\right) = 8\pi G \tag{13}$$

Solving this equation for k we obtain that:

$$k = \frac{1 + 32\pi G(\alpha - \beta\rho)}{12(\alpha - \beta\rho)} - \frac{\sqrt{1 + 32\pi G(\alpha - \beta\rho)(32\pi G(\alpha - \beta\rho) - 4)}}{12(\alpha - \beta\rho)} \tag{14}$$

We can verify that taking the limit $\alpha \to 0$ and $\beta \to 0$ we will recover, as would be expected, the value of k for the GR, that is $k = 8\pi G$. We can observe that in this generalized theory the constant k is dependent not only on the parameters associated with the theory, α and β, but also on the mass density ρ. However, for solutions associated with the vacuum, the mass density is zero, consequently k in the generalized Rastall's gravity coincides with k in GR in this case.

D. Vacuum solution with spherical symmetry

At this point, we are interested in solutions of the field equations that represent static spherically symmetric space-times in generalized Rastall's gravity. In the first place, we consider the trace of the field equation (5) in vacuum: $R(-1 + 4k\alpha) = 0$, this equation is satisfied either by setting $R = 0$ or $\alpha = 1/4k$. In the first case, it is possible to show that the spherical symmetric solution is completely equivalent to the usual Schwarzschild solution in GR. In the second case, we see that the vacuum version of Eq. (5) reads

$$R^{\nu}{}_{\mu} - \frac{1}{4}\delta^{\nu}{}_{\mu}R = 0. \tag{15}$$

The metric on the symmetry of interest, can be written in the usual form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{16}$$

where $B(r)$ and $A(r)$ are functions that are determined using the field equation. In this way, if one uses the metric (16) in the field equation (15), one obtain the following vacuum equations

$$R_{tt} - g_{tt}\frac{1}{4}R = 0, \tag{17}$$

$$R_{rr} - g_{rr}\frac{1}{4}R = 0, \tag{18}$$

$$R_{\theta\theta} - g_{\theta\theta}\frac{1}{4}R = 0, \tag{19}$$

where

$$R_{tt} = -\frac{1}{4}\left(\frac{B'A'}{A^2} + \frac{B'^2}{AB}\right) + \frac{1}{2}\frac{B''}{A} + \frac{B'}{rA}, \tag{20}$$

$$R_{rr} = \frac{1}{4}\left(\frac{B'A'}{BA} + \frac{B'^2}{B^2}\right) - \frac{1}{2}\frac{B''}{B} + \frac{A'}{rA}, \tag{21}$$

$$R_{\theta\theta} = \frac{1}{2}\frac{rA'}{A^2} - \frac{1}{2}\frac{B'r}{BA} + 1 - \frac{1}{A}. \tag{22}$$

Adding equations (17) and (18) and using the previous equations, we obtain the differential relation

$$\frac{A'}{A} + \frac{B'}{B} = 0. \tag{23}$$

The solution of this equation gives a relation between A and B in the form

$$A(r) = B(r)^{-1}. \tag{24}$$

Finally, using this result in Eq. (22) and solving the resulting differential equation, we obtain the solution with spherical symmetry in generalized Rastall's gravity

$$B(r) = 1 + \frac{C}{r} + Dr^2, \tag{25}$$

where C and D are integration constants. In this way, the space-time metric which represents a spherically symmetrical vacuum solution of the generalized Rastall's gravity can finally be written as

$$ds^2 = -(1 + \frac{C}{r} + Dr^2)dt^2 + (1 + \frac{C}{r} + Dr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{26}$$

As this metric does not explicitly depend on β, this solution is equivalent to the space-time obtained in the usual Rastall's theory [6]. In the case of $C = -2GM$ and $D = -\Lambda/3$ this solution can be identified with the Schwarzschild-de Sitter in GR or Schwarzschild-Anti de Sitter for $\Lambda < 0$.

III. NEUTRON STARS

The study of neutron stars is interesting from a nuclear physics point of view, thanks to the extremely dense matter and possible phase transitions inside the stars, as well as a good test for alternative theories of gravity, due to the intense gravitational field created by this object.

Therefore, in order to test the generalized Rastall's gravity, in the next section we derive the equations that describe neutron stars within this theory.

A. Internal solution

Here we present the solution of the modified Einstein equations for the interior of a compact, static and spherically symmetric object.

The distribution of matter inside the star can be described by the energy-momentum tensor of a perfect fluid, given by the following expression:

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu, \tag{27}$$

where p and ρ are respectively the pressure and the energy density of the stellar matter, and U_μ is the 4-velocity of the fluid element, which satisfies $U_\mu U^\mu = -1$.

Now we can work on Eq. (7) together with the metric, Eq. (16), and Eq. (27) to obtain the components of the modified Einstein field equations

$$-\frac{B}{r^2 A} + \frac{B}{r^2} + \frac{A'B}{rA^2} = 8\pi GB\bar{\rho}, \tag{28}$$

$$-\frac{A}{r^2} + \frac{B'}{rB} + \frac{1}{r^2} = 8\pi GA\bar{p}, \tag{29}$$

$$-\frac{B'^2 r^2}{4AB^2} - \frac{A'B'r^2}{4A^2B} + \frac{B''r^2}{2AB} - \frac{A'r}{2A^2} + \frac{B'r}{2AB} = 8\pi Gr^2\bar{p}, \tag{30}$$

where the energy and pressure read

$$\bar{\rho} = \frac{k}{8\pi G}\left[\rho + \frac{T}{4 - \frac{1}{k(\alpha+\beta T)}}\right], \tag{31}$$

$$\bar{p} = \frac{k}{8\pi G}\left[p - \frac{T}{4 - \frac{1}{k(\alpha+\beta T)}}\right], \tag{32}$$

with $T = 3p - \rho$. Note that k in the above equations is given by Eq. (14), where we can see the effect of generalized Rastall's gravity.

From Eq. (28) we can integrate A,

$$A(r) = \left[1 - \frac{2GM(r)}{r}\right]^{-1}, \tag{33}$$

and $M(r)$ is the mass included in the radial coordinate r. The definition of the mass term is

$$M(r) = \int_0^R 4\pi r'^2 \bar{\rho}(r')dr', \tag{34}$$

where R is the radius of the star, which is defined as the radial coordinate at which the pressure vanishes, *i.e.*, $R \equiv r'(p=0)$. Therefore, the total gravitational mass of the neutron stars is $M \equiv M(R)$.

We want to analyze the mass and radius of neutron stars using the pressure and energy density of the nuclear matter inside the star as inputs. The mass equation (34) is one of our equations, and the second one we obtain from a combination of Eqs. (29) and (33) to complete our system:

$$\frac{B'}{2B} = \frac{GM(r)}{r^2}\left[1 + \frac{4\pi r^3\bar{p}}{M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1}. \tag{35}$$

The generalized Rastall's gravity directly affects the energy-momentum conservation, as explained in the previous section. Therefore, from the non-conservation of $T^{\nu}{}_{\mu}$ given by Eq. (3), we obtain

$$\frac{B'}{2B} = -\frac{\bar{p}'}{\bar{p}+\bar{\rho}}. \tag{36}$$

We manipulate the last two equations to obtain the following relation:

$$\bar{p}' = -\frac{GM(r)\bar{\rho}}{r^2}\left[1 + \frac{\bar{p}}{\bar{\rho}}\right]\left[1 + \frac{4\pi r^3\bar{p}}{M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1}. \tag{37}$$

Equations (34) and (37) are the equivalent of the Tolman-Oppenheimer-Volkoff (TOV) [33, 34] equations in the generalized Rastall's gravity. In the next section we use these two equations together with the nuclear equation of state to obtain the mass and radius of a family of neutron star in the context of the modified theory of gravity presented in this work.

B. Numerical results

We are now in a position that allows us to use the equations obtained in the previous section together with realistic equations of state to model neutron stars.

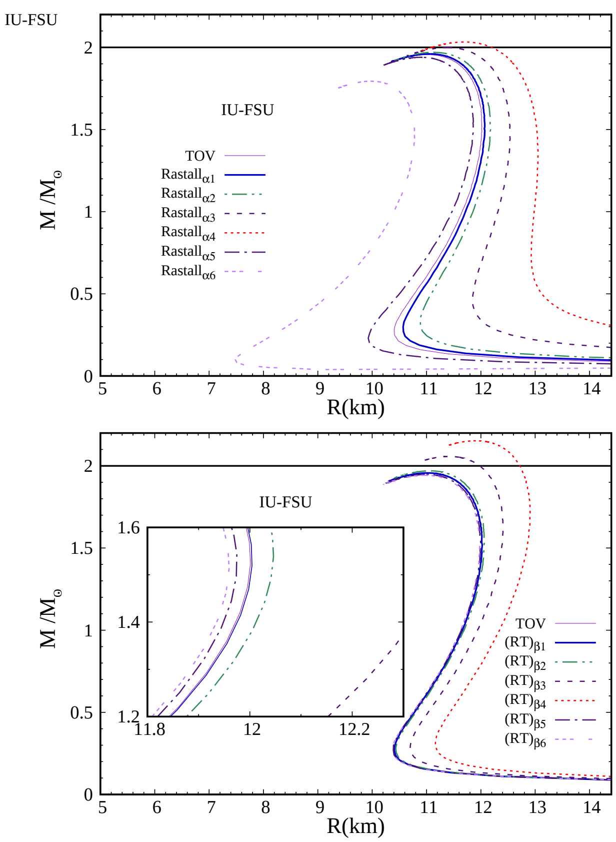

FIG. 1. Mass-radius relation for a family of hadronic stars described with the IU-FSU EoS. We analyze the effects caused by varying the parameter α (top) while keeping the parameter β null and the effects of varying the parameter β (bottom) while keeping α null.

As input to the stellar equilibrium equations, we use two realistic equations of state (EoS) obtained from a relativistic mean field (RMF) approach. We first consider the IU-FSU [35] parametrization because it is able to explain reasonably well both nuclear [36] and stellar matter properties [37]. We then compare the IU-FSU results with the ones obtained with a stiffer EoS calculated with the TM1 parametrization [38]. It is well known that a stiffer EoS leads to a bigger NS maximum mass in contrast to a softer one. For the neutron star crust, we use the well known BPS [39] EoS that describes well the low density region.

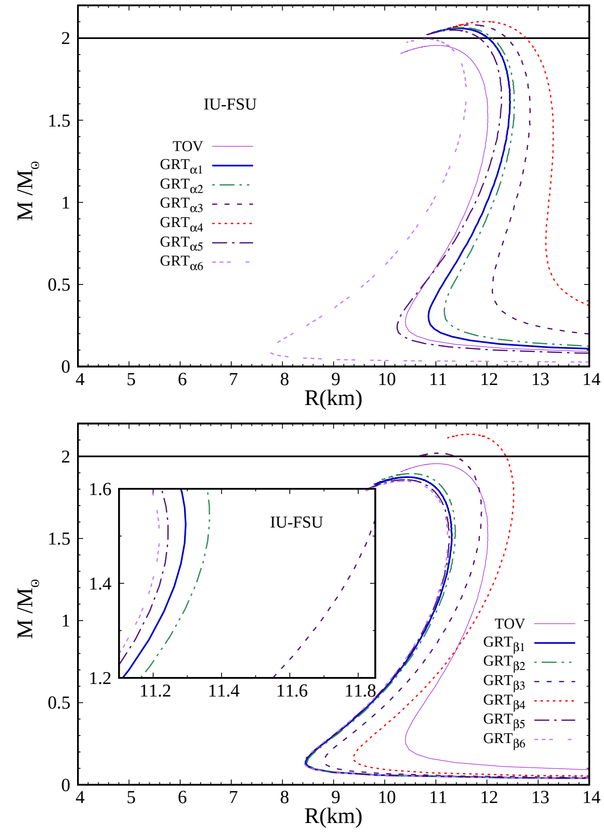

FIG. 2. Mass-radius relation for a family of hadronic stars described with the IU-FSU EoS. We analyze the effects caused by varying the parameter α (top) while keeping the parameter β fixed and the effects of varying the parameter β (bottom) while keeping α fixed.

The differential equations (34) and (37) for the stellar structure can be integrated numerically for the three unknown functions m, p and ρ. Note that this integration occurs from the center to its surface, which is characterized by a point where p vanishes. From different values of the EoS input central density, ρ_c, and from the generalized Rastall's parameters, α and β, we construct the macroscopic properties, i.e., the values of mass and corresponding radius for a family of neutron stars. The results are shown below in the tables and in the corresponding figures (mass-radius profiles). The solutions for the standard case of GR are obtained numerically by using $\alpha = 0$ and $\beta = 0$. They are represented by continuous purple lines in the figures and the resulting values for the maximum mass and the corresponding radius for this solution are listed in the Tables.

In Fig. 1, the effects of both parameters appearing in the generalized Rastall's gravity are individually analyzed for the IU-FSU EoS. In the top panel, we have the results corresponding to the Rastall's gravity solution, *i.e.*, $\alpha \neq 0$ and $\beta = 0$. We obtain the highest (lowest) maximum mass for the most negative (positive) values of α within the range shown in Table I. The radius of the canonical NS ($M = 1.4M_\odot$) is considerably affected. Note a bigger (smaller) radius for the most negative (positive) values of the Rastall parameter. Recent results for neutron stars in the context of Rastall's theory can be found in Ref. [4, 5]. We have checked that although the Rastall's gravity alone affects very little the maximum stellar mass, it considerably increases the corresponding radius, while the canonical radius of the star also increases [4]. On the other hand, the authors in [5] have shown that it is possible to cause the maximum stellar mass to increase at the same time that the canonical radius of the star decreases, however, at the expense of adding a parameter of a second theory. In contrast, we show in our results that regardless of the two EoS tested, we reproduce results similar to those present in [5], however, within the same theory of gravity. In the bottom panel of Fig. 1, we investigate the effects of the RT term alone, *i.e.*, $\alpha = 0$ and $\beta \neq 0$. Note that, for some β values, the effect is bigger on the maximum mass than the the effect of the α parameter in general. As β grows we are able to reproduce more massive neutron stars, keeping the Rastall parameter fixed. In this case, the radius of the canonical NS increases with the increase of the β parameter. These results show that the NS profiles are very sensitive to variations of both parameters.

In Fig. 2, we analyze the effect of having both parameters different from zero, still for the IU-FSU EoS. On the top panel, we fix $\beta = 1 \times 10^{-3}$ and vary α. We do not see a big effect of this parameter on the NS maximum mass, but the importance of the Rastall parameter is clearly seen on the NS radius of the whole family of stars. As in the previous case, the most positive α gives the smaller radius. Since we are now in the complete generalized Rastall's case, it is interesting to compare this result with the TOV results, noticing that the α_6 curve on this plot gives a smaller canonical radius and bigger maximum mass to a NS fed with the same nuclear inputs. On the bottom panel, we fix $\alpha = 1.3 \times 10^{-3}$ and vary β. We obtain a maximum increase (decrease) of both NS radius and maximum mass for the most positive (negative) parameter. Here we call attention to curve β_3 which achieves a maximum mass bigger than $2M_\odot$ in accordance with [40, 41], together with a small $1.4M_\odot$ radius, which lies inside the range of the recent work [42], where by combining data from multi-messenger observations and nuclear physics, the authors obtained the most stringent constraint to the canonical neutron star radius, $R_{1.4} = 11.0^{+0.9}_{-0.6}$ km.

In Fig. 3 and 4 we repeat the analyses now with a stiffer EoS, TM1. The general effects of the α and β parameters are the same as in the IU-FSU EoS. It is interesting to notice that TM1 shows a bigger canonical radius than the IU-FSU; however, note in the left panels of Fig. 3 and 4 that the generalized Rastall's gravity gives a ≈1 km smaller radius for the NS of $M = 1.4M_\odot$ while keeping the maximum mass above the required $2M_\odot$.

It is important to remark that the use of the generalized Rastall's theory yields similar variations of the macroscopic quantities (as compared with the used of the TOV equations) which are independent of the chosen nuclear EoS (within the two representative ones analysed in this work). Moreover, although the generalized version of the Rastall's gravity allows more flexibility in the calculation of the macroscopic stellar properties due to the inclusion of two independent parameters, it does not fix existing caveats of the EoS. Hence, an EoS that satisfies bulk nuclear matter properties is still required as input to the generalized Rastall's equations.

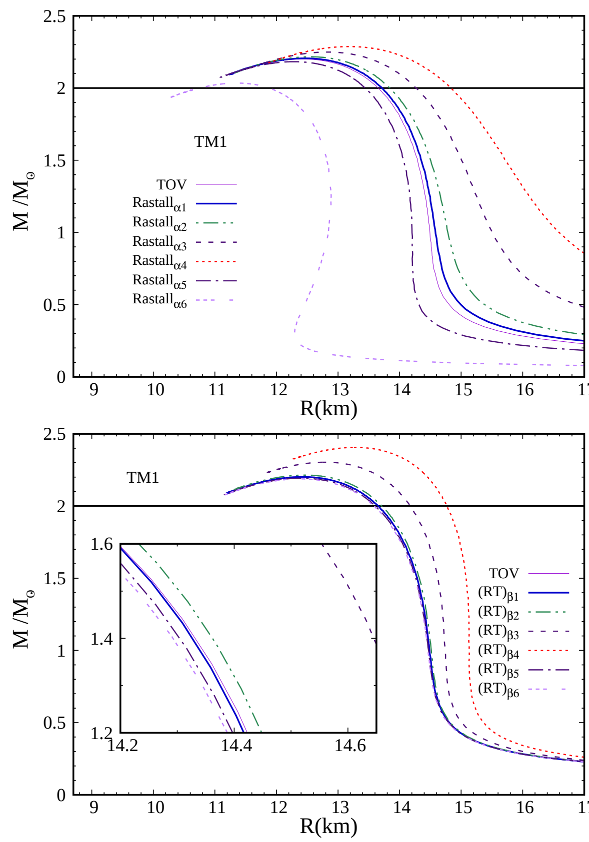

FIG. 3. Mass-radius relation for a family of hadronic stars described with the TM1 EoS. We analyze the effects caused by varying the parameter α (top) while keeping the parameter β null and the effects of varying the parameter β (bottom) while keeping α null.

IV. CONCLUSIONS

In this work we have generalized Rastall's theory of gravity. Original Rastall's gravity breaks the energy-momentum conservation making $T^{\mu}{}_{\nu;\,\mu} = A_\nu = \lambda R_{,\,\nu}$, where a dependence on the curvature R appears on the derivative of $T^{\mu}{}_{\nu;\,\mu}$. We propose that this derivative also depends on the trace of the energy momentum tensor, T, *i.e.* the function A_ν is given now by

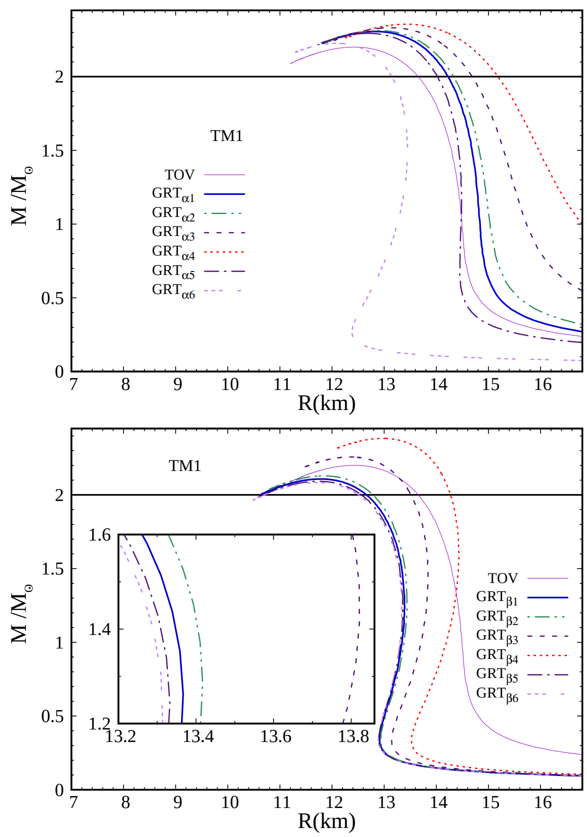

FIG. 4. Mass-radius relation for a family NS within the TM1 EoS. We analyze the effects caused by varying the parameter α (top) while keeping the parameter β fixed and the effects of varying the parameter β (bottom) while keeping α fixed.

$A_\nu = (\alpha\delta^\mu{}_\nu R + \beta\delta^\mu{}_\nu RT)_{;\,\mu}$. Initially we have obtained the external solution in the case of a space-time with spherical symmetry. We have shown that this solution represents the gravitational field outside a spherical mass, and unlike the GR, the solution in the context of the generalized Rastall's gravity gives two types of space-time solutions, depending on the choice of the trace of the energy-moment tensor. This property of the external solution can be regarded as an intrinsic property of the generalized theory since other choices of breaking conservation law lead to the same result. In addition, we have shown that the gravitational parameter k depends on the energy density in contrast to the constant value $k = 8\pi G$ in general relativity.

We tested the theory in neutron stars using two different RMF EoS as inputs and noted a considerable effect of the alternative gravity theory in the NS mass-radius diagrams. The results presented here show that, with small deviations from the GR case, an important change on the NS profile can be obtained within the same nuclear physics inputs.

The recent result of LIGO-Virgo [43] with a possible NS of $2.6M_\odot$ is also of particular interest. If such massive NS is confirmed by future observations it will be a big challenge for the compact objects community to describe it. From the nuclear physics point of view, one needs a very stiff EoS at high densities to support such a high maximum mass together with a soft EoS at low densities to keep a radius of the order of 11 km for the canonical $1.4M_\odot$ NS. From the gravity point of view one can explore theories beyond GR such as the one examined in the present paper.

In future works, it would be interesting to study gravitational effects in astrophysical and cosmological systems due to the choice of other combinations of R and T. Moreover, the application of the present formalism in anisotropic stars is the next step towards a more realistic description of these compact objects.

V. ACKNOWLEDGMENTS

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TABLE I. Macroscopic properties for different values of the α and β parameters corresponding to the mass-radius diagram in FIG.1.

		TOV	$\text{Rastall}_{\alpha 1}$	$\text{Rastall}_{\alpha 2}$	$\text{Rastall}_{\alpha 3}$	$\text{Rastall}_{\alpha 4}$	$\text{Rastall}_{\alpha 5}$	$\text{Rastall}_{\alpha 6}$
Parameters	$\boldsymbol{\alpha}$	**0.0**	-1×10^{-4}	-3×10^{-4}	-1×10^{-3}	-2×10^{-3}	3×10^{-4}	2×10^{-3}
	β	0.0	0.0	0.0	0.0	0.0	0.0	0.0
IU-FSU	M_{max}	1.95 $M_\odot$	1.96 $M_\odot$	1.96 $M_\odot$	1.99 $M_\odot$	2.03 $M_\odot$	1.93 $M_\odot$	1.79 $M_\odot$
	$R_{1.4}$	12.00 km	12.04 km	12.15 km	12.52 km	13.04 km	11.83 km	10.76 km
	C_{max}	0.177	0.177	0.176	0.175	0.173	0.177	0.179
	$C_{1.4}$	0.116	0.116	0.115	0.111	0.107	0.118	0.130
		General Relativity	Generalized Rastall's gravity					
		TOV	$(\text{RT})_{\beta 1}$	$(\text{RT})_{\beta 2}$	$(\text{RT})_{\beta 3}$	$(\text{RT})_{\beta 4}$	$(\text{RT})_{\beta 5}$	$(\text{RT})_{\beta 6}$
Parameters	$\boldsymbol{\beta}$	**0.0**	1×10^{-5}	1×10^{-4}	1×10^{-3}	3×10^{-3}	-5×10^{-5}	-8×10^{-5}
	α	0.0	0.0	0.0	0.0	0.0	0.0	0.0
IU-FSU	M_{max}	1.95 $M_\odot$	1.95 $M_\odot$	1.96 $M_\odot$	2.05 $M_\odot$	2.15 $M_\odot$	1.94 $M_\odot$	1.94 $M_\odot$
	$R_{1.4}$	12.00 km	12.00 km	12.03 km	12.33 km	12.78 km	11.96 km	11.95 km
	C_{max}	0.177	0.176	0.177	0.179	0.180	0.176	0.176
	$C_{1.4}$	0.116	0.116	0.116	0.113	0.109	0.117	0.117

TABLE II. Macroscopic properties for different values of the α and β parameters corresponding to the mass-radius diagram in FIG.2

		TOV	$\text{GRT}_{\alpha 1}$	$\text{GRT}_{\alpha 2}$	$\text{GRT}_{\alpha 3}$	$\text{GRT}_{\alpha 4}$	$\text{GRT}_{\alpha 5}$	$\text{GRT}_{\alpha 6}$
Parameters	$\boldsymbol{\alpha}$	**0.0**	-1×10^{-4}	-3×10^{-4}	-1×10^{-3}	-2×10^{-3}	3×10^{-4}	2×10^{-3}
	β	0.0	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}
IU-FSU	M_{max}	1.95 $M_\odot$	2.05 $M_\odot$	2.06 $M_\odot$	2.08 $M_\odot$	2.10 $M_\odot$	2.04 $M_\odot$	1.99 $M_\odot$
	$R_{1.4}$	12.00 km	12.38 km	12.48 km	12.81 km	13.30 km	12.20 km	11.47 km
	C_{max}	0.177	0.179	0.179	0.177	0.175	0.180	0.184
	$C_{1.4}$	0.116	0.113	0.112	0.109	0.105	0.114	0.122
		General Relativity	Generalized Rastall's gravity					
		TOV	$\text{GRT}_{\beta 1}$	$\text{GRT}_{\beta 2}$	$\text{GRT}_{\beta 3}$	$\text{GRT}_{\beta 4}$	$\text{GRT}_{\beta 5}$	$\text{GRT}_{\beta 6}$
Parameters	$\boldsymbol{\beta}$	**0.0**	1×10^{-5}	1×10^{-4}	1×10^{-3}	3×10^{-3}	-5×10^{-5}	-8×10^{-5}
	α	0.0	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}
IU-FSU	M_{max}	1.95 $M_\odot$	1.87 $M_\odot$	1.89 $M_\odot$	2.01 $M_\odot$	2.13 $M_\odot$	1.85 $M_\odot$	1.84 $M_\odot$
	$R_{1.4}$	12.00 km	11.28 km	11.34 km	11.78 km	12.33 km	11.23 km	11.21 km
	C_{max}	0.177	0.179	0.179	0.182	0.182	0.178	0.177
	$C_{1.4}$	0.116	0.124	0.123	0.118	0.113	0.124	0.124

TABLE III. Macroscopic properties for different values of the α and β parameters corresponding to the mass-radius diagram in FIG.3.

		TOV	$\text{Rastall}_{\alpha1}$	$\text{Rastall}_{\alpha2}$	$\text{Rastall}_{\alpha3}$	$\text{Rastall}_{\alpha4}$	$\text{Rastall}_{\alpha5}$	$\text{Rastall}_{\alpha6}$
	$\boldsymbol{\alpha}$	**0.0**	-1×10^{-4}	-3×10^{-4}	-1×10^{-3}	-2×10^{-3}	3×10^{-4}	2×10^{-3}
Parameters	β	0.0	0.0	0.0	0.0	0.0	0.0	0.0
TM1	M_{max}	2.20 $M_\odot$	2.20 $M_\odot$	2.21 $M_\odot$	2.24 $M_\odot$	2.28 $M_\odot$	2.18 $M_\odot$	2.03 $M_\odot$
	$R_{1.4}$	14.34 km	14.42 km	14.57 km	15.10 km	15.86 km	14.11 km	12.85 km
	C_{max}	0.177	0.176	0.176	0.174	0.172	0.177	0.178
	$C_{1.4}$	0.097	0.097	0.096	0.092	0.088	0.099	0.108
		General Relativity	Generalized Rastall's gravity					
		TOV	$(\text{RT})_{\beta1}$	$(\text{RT})_{\beta2}$	$(\text{RT})_{\beta3}$	$(\text{RT})_{\beta4}$	$(\text{RT})_{\beta5}$	$(\text{RT})_{\beta6}$
	$\boldsymbol{\beta}$	**0.0**	1×10^{-5}	1×10^{-4}	1×10^{-3}	3×10^{-3}	-5×10^{-5}	-8×10^{-5}
Parameters	α	0.0	0.0	0.0	0.0	0.0	0.0	0.0
TM1	M_{max}	2.20 $M_\odot$	2.20 $M_\odot$	2.21 $M_\odot$	2.30 $M_\odot$	2.40 $M_\odot$	2.19 $M_\odot$	2.18 $M_\odot$
	$R_{1.4}$	14.34 km	12.41 km	12.46 km	12.78 km	13.27 km	12.39 km	12.38 km
	C_{max}	0.177	0.177	0.177	0.179	0.180	0.176	0.176
	$C_{1.4}$	0.097	0.097	0.097	0.095	0.092	0.097	0.097

TABLE IV. Macroscopic properties for different values of the α and β parameters corresponding to the mass-radius diagram in FIG.4

		TOV	$\text{GRT}_{\alpha1}$	$\text{GRT}_{\alpha2}$	$\text{GRT}_{\alpha3}$	$\text{GRT}_{\alpha4}$	$\text{GRT}_{\alpha5}$	$\text{GRT}_{\alpha6}$
	$\boldsymbol{\alpha}$	**0.0**	-1×10^{-4}	-3×10^{-4}	-1×10^{-3}	-2×10^{-3}	3×10^{-4}	2×10^{-3}
Parameters	β	0.0	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}	1×10^{-3}
TM1	M_{max}	2.20 $M_\odot$	2.30 $M_\odot$	2.31 $M_\odot$	2.33 $M_\odot$	2.35 $M_\odot$	2.29 $M_\odot$	2.22 $M_\odot$
	$R_{1.4}$	14.34 km	14.72 km	14.87 km	15.37 km	16.10 km	14.45 km	13.43 km
	C_{max}	0.177	0.179	0.179	0.177	0.175	0.180	0.183
	$C_{1.4}$	0.097	0.095	0.094	0.091	0.086	0.096	0.104
		General Relativity	Generalized Rastall's gravity					
		TOV	$\text{GRT}_{\beta1}$	$\text{GRT}_{\beta2}$	$\text{GRT}_{\beta3}$	$\text{GRT}_{\beta4}$	$\text{GRT}_{\beta5}$	$\text{GRT}_{\beta6}$
	$\boldsymbol{\beta}$	**0.0**	1×10^{-5}	1×10^{-4}	1×10^{-3}	3×10^{-3}	-5×10^{-5}	-8×10^{-5}
Parameters	α	0.0	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	1.3×10^{-3}
TM1	M_{max}	2.20 $M_\odot$	2.10 $M_\odot$	2.12 $M_\odot$	2.25 $M_\odot$	2.38 $M_\odot$	2.09 $M_\odot$	2.08 $M_\odot$
	$R_{1.4}$	14.34 km	13.36 km	13.41 km	13.83 km	14.40 km	13.33 km	13.31 km
	C_{max}	0.177	0.178	0.179	0.182	0.183	0.178	0.177
	$C_{1.4}$	0.097	0.104	0.104	0.101	0.097	0.105	0.105