Conserved charges in general relativity

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We present a precise definition of a conserved quantity from an arbitrary covariantly conserved current available in a general curved spacetime. This definition enables us to define energy and momentum for matter by the volume integral. As a result we can compute charges of well-known black holes just as an electric charge of an electron in electromagnetism by the volume integration of a delta function singularity. As a byproduct we show that the definition leads to a correction to the known mass formula of a compact star in the Oppenheimer-Volkoff equation. We finally comment on a definition of generators associated with a vector field on a general curved manifold.

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1. INTRODUCTION

Since Einstein submitted papers on general relativity [1], classical or quantum field theory on a curved spacetime has extensively been investigated. When spacetime is curved, the physical quantities defined on flat spacetime are required to be modified suitably in accordance with general covariance. For example, a conserved current, which exists in the presence of global symmetry in the system [2], is modified to be a covariantly conserved one on a general curved spacetime.

However there has been no general argument to define a conserved charge from a covariantly conserved current, which inevitably causes a problem to define energy and momentum. Einstein originally argued that the conservation law of energy and momentum for matter follows as long as they are combined with those for gravitational field [1]. (See also [3].) The corresponding energy momentum tensor for the gravitational field, however, is not covariant under general coordinate transformation. As a result an energy defined as in the case of flat spacetime depends on a coordinate system and conserves only in the particular frame.

One way to circumvent this issue is to define an energy locally on the asymptotic region of spacetime called quasi-local energy. Initially the quasi-local energy and momentum were studied on an asymptotically flat spacetime by recasting gravity system into the Hamiltonian dynamics known as the ADM formalism [4]. (See also [5].) They are defined by a surface integral in the asymptotic region, by which the invariance under a class of general coordinate transformations preserving a boundary condition was achieved. This result has been further extended for a more general curved spacetime with surface terms suitably incorporated [6–10]. A caveat in this extension is that boundary terms accompany with divergence even in the flat spacetime, so that one needs to subtract it by comparing a reference frame or by adding local counter terms.

The authors of the present letter investigated a property of a black hole holographically realized by the flow equation method [11]. In the study we encountered a situation to evaluate the energy of the total system with matter spread all over the space, which is required to be evaluated by the volume integral of the energy density. We discovered a definition of a conserved charge from a covariantly conserved current in a general curved spacetime, which improves the one given in [12, 13] for special backgrounds. This allows us to define energy and momentum for matter in a form of the volume integral at an arbitrary time slice of a given curved spacetime. A virtue is to enable one to evaluate charges of black holes just like an electric charge of the electron in electromagnetism by the volume integral of the delta function singularity. As a byproduct of our approach, we apply our definition to the mass of a compact star and discover that there is a correction to the mass formula obtained from the Oppenheimer-Volkoff equation.

2. CONSERVED CHARGE FROM COVARIANTLY CONSERVED CURRENT

Consider any classical or quantum field theory on a general curved spacetime. Suppose there exists a covariantly conserved current J^μ, $\nabla_\mu J^\mu = 0$, where ∇_μ is the covariant derivative for the metric $g_{\mu\nu}$. Then we claim that the following quantity is conserved under the given time evolution

$$Q(t) := \int_{M_t} d^{d-1}\vec{x}\, \sqrt{|g|} J^0(t, \vec{x}), \tag{1}$$

where M_t represents a time slice of the spacetime M at the time t, g denotes the determinant of $g_{\mu\nu}$, and d is the dimension of the spacetime M. If there exists boundary for M_t, we set the boundary condition for the fields to fall off sufficiently fast at boundary of M_t for all t. We emphasize that g is the determinant of the metric in the total spacetime, which contains the time components.

To show this, we assume the spacetime has the foliation structure for simplicity. (The same argument is used in literature. For example, see [14].) Let us consider the same quantity defined by (1) at another time slice with t' greater than t, and take a submanifold M' with the foliation structure whose boundary contains M_t and $M_{t'}$. Such a manifold may be written formally as $M' = \prod_{t\leq s\leq t'} M_s$. Under the boundary condition, the difference between $Q(t')$ and $Q(t)$ becomes

$$Q(t') - Q(t) = \int_{M'} d^d x\, \partial_\mu(\sqrt{|g|}J^\mu(t,\vec{x})) = 0, \qquad (2)$$

where we use $\partial_\mu(\sqrt{|g|}J^\mu(t,\vec{x})) = \sqrt{|g|}\nabla_\mu J^\mu(t,\vec{x}) = 0$. This proves that $Q(t)$ is independent of t. The charge Q is a scalar under the assumption, though the generalization is straightforward. Note that Q is not a scalar if it is defined from a higher rank tensor rather than the vector.

This formula can be applied to the computation of a conserved charge for any gravitational systems with a Killing vector. A covariantly conserved current associated with a Killing vector ξ can be constructed as

$$J^\mu = T^\mu{}_\nu \xi^\nu, \qquad (3)$$

where $T^\mu{}_\nu$ is the given energy momentum tensor for matter. It can be easily shown that this is covariantly conserved by using $\nabla_\mu T^\mu{}_\nu = 0$ and $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$ [13, 15, 16]. In the next section we compute charges of several black holes by using this formula.

3. CONSERVED CHARGES OF BLACK HOLES

In this section we compute a conserved charge for well-known black holes employing the presented formula.

3-1. Schwarzschild black hole

In order to explain a key idea of the calculation we start with the simplest setup. That is, we begin with the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \qquad (4)$$

This is satisfied by the Schwarzshild black hole solution:

$$ds^2 = -f(r)(dx^0)^2 + \frac{1}{f(r)}dr^2 + r^2\tilde{g}_{ij}dx^i dx^j, \qquad (5)$$

where r is the radial coordinate and the $d-2$ dimensional manifold fibered over the cone is an Einstein manifold, whose Ricci tensor is given by ${}^{(d-2)}R_{ij} = (d-3)k\tilde{g}_{ij}$ with a constant k, and

$$f(r) = \frac{-2\Lambda r^2}{(d-2)(d-1)} + k - \frac{2G_N M}{r^{d-3}}. \qquad (6)$$

Note that for a positive or non-positive k the submanifold is compact or non-compact, respectively.

Since this is a static solution, there exists a Killing vector with $\xi^\mu = -\delta^\mu_0$, which corresponds to the time translation. Thus the corresponding charge is the energy of the system:

$$E = \int d^{d-1}\vec{x}\,\sqrt{|g|}(-T^0{}_0), \qquad (7)$$

where the matter energy momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{8\pi G_N}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}), \qquad (8)$$

with G_N the Newton constant. According to the equation of motion (4) this energy momentum tensor seems to vanish on shell, but it does not. We emphasize that it vanishes except a singularity located at $r=0$. This singularity contributes to the charge.

In order to compute the contribution, we expand the stress tensor perturbatively around infinity to extract a pole. This can be done by separating the metric into the regular part $\bar{g}_{\mu\nu}$ and the singular part $h_{\mu\nu}$, the latter of which is given by

$$h_{\mu\nu}dx^\mu dx^\nu = -\delta f(dx^0)^2 + \left(\frac{1}{f} - \frac{1}{f-\delta f}\right)dr^2, \quad (9)$$

where $\delta f = -2M/r^{d-3}$. At the leading order, we have

$$T^0{}_0 = -\frac{1}{2}\left(\frac{1}{\sqrt{|\bar{g}|}}\partial_\mu(\sqrt{|\bar{g}|}\bar{g}^{\mu\nu}\partial_\nu h^0_0) - \bar{f}^{-1}\bar{f}'^2 h^0_0\right) + \bar{\nabla}^0\bar{\nabla}_\sigma h^\sigma_0 - h^0_0 - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\sigma h^{\sigma\mu} + \cdots, \qquad (10)$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to the metric $\bar{g}_{\mu\nu}$, $\bar{f} := f - \delta f$, and the ellipsis represents the higher order terms of h. This must vanish except at the origin, and indeed this can be written as

$$T^0{}_0 = \frac{d-2}{16\pi G_N r^{d-2}}\partial_r\left(r^{d-3}\delta f\right) + \cdots, \qquad (11)$$

which has the desired property. Plugging this into (7) we can compute the charge as

$$E = -\int d^{d-1}\vec{x}\,\sqrt{|\tilde{g}|}\frac{d-2}{16\pi G_N}\partial_r\left(r^{d-3}\delta f\right) = \rho V_{d-2}, \quad (12)$$

where $V_{d-2} = \int d^{d-2}x\sqrt{|\tilde{g}|}$ is the volume of the Einstein manifold with $\tilde{g}$ being the determinant of $\tilde{g}_{ij}$, and $\rho = (d-2)M/(8\pi)$ is the energy or mass density. To evaluate r integral we employ the Stokes's theorem. Note that the higher order terms do not contribute to the surface integral. Our result reproduces the known result obtained by other methods. (For example, see eq.(2.5) in Ref. [8].)

The first term in Eq. (11) is indeed exact. This can be seen by calculating $T^0{}_0$ directly from the Ricci tensor, and more insightfully, it can be written in a form proportional to a delta function:

$$T^0{}_0 = \frac{d-2}{16\pi G_N r^{d-2}}\partial_r\left(r^{d-3}F\right) = -\rho\frac{\delta(r)}{r^{d-2}}. \qquad (13)$$

We here insert the step function θ for the singular term in (6), so that $F(r) = \delta f(r)\theta(r)$ and $\delta(r) = \frac{d\theta(r)}{dr}$. Thus the matter energy momentum tensor $T^{\mu}{}_{\nu}$ for the black hole can be understood as a distribution. We can calculate E as the volume integral, which also leads to (12), justifying the use of the step function to handle the singularity at $r = 0$.

3-2. Reissner-Nordström black hole

Below we present a more illuminative computation of a mass of a charged black hole in general d dimensions, whose metric is given in (5) by replacing $f(r)$ with $f_q(r) = f(r) + \frac{d-3}{d-2} 8\pi G_N q^2 r^{-2(d-3)}$, together with the gauge potential $A_\mu = (-\frac{q}{r^{d-3}} + \frac{q}{r_+^{d-3}})\delta^0_\mu$, where q, r_+ are constants[17]. This configuration of gravitational and gauge fields satisfies the equations of motion given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N (T^G_{\mu\nu} + T^A_{\mu\nu}),\ \nabla_\mu F^{\mu}{}_{\nu} = J_\nu, \quad (14)$$

where $F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and $T^A_{\mu\nu} := F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$. Here $T^G_{\mu\nu}$ and J_ν explicitly represent the singular contributions of the metric and the gauge potential at $r = 0$, respectively. Explicitly $(T^G + T^A)^0{}_0$ is given in (13) by replacing F with $F_q = F + \frac{d-3}{d-2} 8\pi G_N q^2 r^{-2(d-3)}$.

Since this metric is also static, the energy defined by (7) is conserved. However this charge diverges, due to the contribution of the electromagnetic field. Physically, this divergence can be interpreted as the self-energy for the charged point particle. Indeed it remains even for the flat space-time with $M = 0$ and $\Lambda = 0$. Classically, the charged black hole has the infinite energy due to the infinite electromagnetic energy. Thus the renormalization as well as the quantization of the gauge field on the curved space are needed to fix this problem, as was so on the flat space.

Fortunately, since $\nabla_\mu (T^G)^\mu{}_0 = 0$ (thus $\nabla_\mu (T^A)^\mu{}_0 = 0$), we can define an energy from the covariantly conserved T^G alone without electromagnetic energy, and $(T^G)^0{}_0$ is identically given by (13). We thus obtain

$$E_G = \int d^{d-2}\vec{x} \int dr \sqrt{|g|} (T_G)^0{}_0 \xi^0 = V_{d-2}\rho, \quad (15)$$

which reproduces the result in the special case of [17].

This system allows another conserved quantity, thanks to the invariance under the $U(1)$ gauge transformation by $\delta A_\mu = \partial_\mu \theta$, which leads to

$$\partial_\mu j^\mu = 0, \quad j^\mu = \nabla_\nu \left(\sqrt{|g|} F^{\mu\nu} \right) \quad (16)$$

without using the Maxwell equation. According to our prescription, $Q_c = \int d^{d-2}x \int dr \sqrt{|g|} J^0$ with $J^0 = j^0/\sqrt{|g|}$ gives the conserved electric charge, which is evaluated as $Q_c = V_{d-2}(d-3)q$. At $d = 4$ for $k > 0$, $Q_c = 4\pi q$.

3-3. BTZ black hole

As a final example, we compute a charge different from a mass. To this end we consider a BTZ black hole and compute its angular momentum [18]. The metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\phi - \omega(r)dt)^2, \quad (17)$$

where

$$f(r) = \frac{r^2}{L^2} - 2G_N M\theta(r) + \frac{G_N^2 J^2}{4r^2}, \quad \omega(r) = \frac{G_N J}{2r^2}, \quad (18)$$

with M, J are constants, satisfies the Einstein equation in three dimensions. We insert the step function to the constant part to emphasize that this solution is valid except the origin.

This BTZ black hole has not only a Killing vector with respect to the time translation but also the one which rotates the system, $\xi^\mu = \delta^\mu_\phi$. As in the previous cases the first one defines the mass, which can be similarly computed as $E = \frac{M}{4}$. On the other hand, the second Killing vector define an angular momentum:

$$P_\phi = \int d^2x \sqrt{|g|} T^0{}_\phi. \quad (19)$$

$T^0{}_\phi$ is computed from the Einstein tensor as $T^0{}_\phi = -\frac{1}{16\pi G_N r} \partial_r \left(r^3 \omega'(r) \right)$. Thus we find $P_\phi = \frac{J}{8}$, which reproduces the known result [18].

4. MASS OF A COMPACT STAR

Our formula for the conserved charge leads to nontrivial corrections to a mass of a compact star.

4-1. Oppenheimer-Volkoff equation

Let us consider the energy momentum tensor for the fluid, given by

$$T^0{}_0 = -\rho(r), \quad T^r{}_r = P(r), \quad T^i_j = \delta^i_j P(r), \quad (20)$$

where $\rho(r)$ is the energy density and $P(r)$ is the pressure. The Oppenheimer-Volkoff equation[19, 20] for the metric eq. (5) with $1/f(r)$ in the second term replaced by another function $h(r)$ becomes

$$-\frac{dP(r)}{dr} = \frac{G_N \rho(r) M(r)}{r^{d-2}} \left(1 + \frac{P(r)}{\rho(r)} \right) h(r)$$
$$\times \left\{ d - 3 + \frac{r^{d-1}}{(d-2)M(r)} \left(8\pi P(r) - \frac{2\Lambda}{(d-1)G_N} \right) \right\}, (21)$$

where

$$\frac{1}{h(r)} = \frac{-2\Lambda r^2}{(d-2)(d-1)} + k - \frac{2G_N M(r)}{r^{d-3}}, \quad (22)$$

and

$$M(r) = \frac{8\pi}{d-2}\int_0^r ds\, s^{d-2}\rho(s), \quad M(0) = 0. \tag{23}$$

At the surface of the star, the pressure vanishes, $P(r = R) = 0$, where R is the radius of the star, and the energy momentum tensor is covariantly conserved, $\nabla_\mu T^\mu{}_\nu(R) = 0$. Since $P(r) = \rho(r) = 0$ at $r > R$ (outside the star), the metric becomes the Schwarzschild, namely $f(r) = 1/h(r)$ with the mass parameter $M = M(R)$ in (5).

4-2. Mass of a compact star

Our definition leads to the conserved total energy of this system for the Killing vector $\xi^\mu = -\delta^\mu_0$ as

$$\begin{aligned} E &= \int d^{d-2}\vec{x}\int_0^\infty dr\,\sqrt{-g(r)}\,T^0{}_0(r)\xi^0(r) \\ &= V_{d-2}\int_0^R dr\sqrt{f(r)h(r)}r^{d-2}\rho(r). \end{aligned} \tag{24}$$

This total mass is different from the Schwarzschild mass parameter. Indeed

$$\begin{aligned} \frac{8\pi}{(d-2)V_{d-2}}\times E &= \int_0^R dr\,\sqrt{f(r)h(r)}\frac{dM(r)}{dr} \\ &= M(R) - \int_0^R dr\,\frac{M(r)}{2}\frac{d}{dr}\log[f(r)h(r)]. \end{aligned} \tag{25}$$

Therefore, an estimation for the compact star mass including its maximum mass should be replaced with the above, though a size of the correction might be small. (25) also tells us that the total mass of the star can not be written as a surface term (the 1st term) alone and the volume integral (the 2nd term) is necessary. Furthermore, the $R \to 0$ limit of (25) formally reduces to the black hole mass.

5. DISCUSSION

In this letter, we have proposed a general definition of a conserved charge from any covariantly conserved current, which requires no specific asymptotic behaviors/approximations for the metric, or no subtraction of boundary contributions. Our definition has reproduced the mass, electric charge and angular momentum of black holes known in the literature. Since the covariant formula requires the non-zero energy momentum tensor to define the mass, it is clear that the black hole has non-zero energy momentum tensor at $r = 0$, like the charged point particle in the classical electrodynamics. We have also demonstrated that the mass of the compact star defined as the conserved charge cannot be written as a surface term alone, so that it differs from the Schwarzschild mass employed in literature.

Our proposal is quite generic, so we expect plenty of applications in future. As such a potential application, we consider a more general case where any Killing vectors do not exist. We can still consider a charge or a generator associated with a general vector field ξ^μ as

$$Q[\xi](t) = \int_{M_t} d^{d-1}\vec{x}\,\sqrt{|g|}T^0{}_\nu\xi^\nu. \tag{26}$$

Using the similar argument before, we obtain

$$\begin{aligned} \frac{dQ[\xi]}{dt} &= \int_{M_t} d^{d-1}\vec{x}\sqrt{|g|}\rho(x), \\ \rho(x) &:= \frac{(G^{\mu\nu} + \Lambda g^{\mu\nu})}{16\pi G_N}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) \end{aligned} \tag{27}$$

where $\rho = 0$ if ξ is a Killing vector. A change of the charge $Q[\xi]$ can be calculated by the volume integral of ρ, expressed in terms of the gravitational field through the Einstein equation. (27) may give a hint for a *generic conservation equation* in general relativity. This argument will hold not only for a Lorentzian manifold but also for a more general one. We will return to this interesting problem in future studies.

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