QUANTUM BAYESIAN NETS

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Published in: Int. Jour. Of Mod. Phys. B9(1995)295-337
The principles of this paper have been implemented
in commercial software available at
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ABSTRACT

We begin with a review of a well known class of networks, Classical Bayesian (CB) nets (also called causal probabilistic nets by some). Given a situation which includes randomness, CB nets are used to calculate the probabilities of various hypotheses about the situation, conditioned on the available evidence. We introduce a new class of networks, which we call Quantum Bayesian (QB) nets, that generalize CB nets to the quantum mechanical regime. We explain how to use QB nets to calculate quantum mechanical conditional probabilities (in case of either sharp or fuzzy observations), and discuss the connection of QB nets to Feynman Path integrals. We give examples of QB nets that involve a single spin-$\frac{1}{2}$ particle passing through a configuration of two or three Stern-Gerlach magnets. For the examples given, we present the numerical values of various conditional probabilities, as calculated by a general computer program especially written for this purpose.

1. INTRODUCTION

The artificial intelligence and expert systems literature contains a large number of articles ([1]-[8]) discussing the theory and application of Classical Bayesian (CB) nets. At least one software package[9] that implements this theory is commercially available. Amazingly, the physics community, to whom this paper is mainly addressed, seems not to have discovered or used CB nets yet. Therefore, we begin this paper with a review of CB nets. However, the real purpose of this paper is to introduce a new class of nets, which we shall call Quantum Bayesian (QB) nets, that generalize CB nets to the quantum mechanical regime. In this paper, to illustrate QB nets, we use them to predict conditional probabilities for experiments comprising combinations of two or three Stern-Gerlach magnets. In future papers, we will use QB nets to analyze quantum optical experiments and other physical situations. Earlier workers have devised networks for quantum mechanics (for eg., Feynman diagrams, and the "trajectory graphs" of Ref.[10]), but their nets differ substantially from ours.

Henceforth, we will underline random variables. We will write $P(\underline{x} = x)$ for the probability that the random variable $\underline{x}$ assumes the particular value x[11]. Sometimes, if there is no danger of confusion, we will write $P(x)$ rather than $P(\underline{x} = x)$. Similarly, we will often write $P(y|x)$ instead of $P(\underline{y} = y|\underline{x} = x)$. $P(\underline{y} = y|\underline{x} = x)$, the conditional probability that $\underline{y}$ assumes the value y given that $\underline{x}$ assumes the value x, is defined by

$$P(\underline{y} = y|\underline{x} = x) = \frac{P(\underline{y} = y, \underline{x} = x)}{P(\underline{x} = x)} \ . \tag{1.1}$$

Suppose $\underline{h}$ is a random variable that can take on as values each of a number of *hypotheses*, and $\underline{e}$ is a random variable representing the observations that constitute the available *evidence*. We often want to calculate the posterior probability $P(h|e)$ in terms of the prior probability $P(h)$. To do this, one may use Bayes's rule, which says

$$P(h|e) = \frac{P(e|h)P(h)}{\sum_{h'} P(e|h')P(h')} \ . \tag{1.2}$$

Equation(1.2) is a simple consequence of Eq.(1.1). The subject of CB nets may be viewed as an extension of Bayes's rule.

A CB net has two parts: a diagram consisting of nodes with arrows connecting some pairs of these nodes, and a collection of probabilities, one per node. For example, the digital circuits of electrical engineering can be modelled as CB nets if each NAND (or, alternatively, each AND, OR and NOT gate) is replaced by a node, and each connecting cable is replaced by an arrow pointing in the direction of current flow. Of course, in the usual digital circuits, the NAND gates are deterministic, their output being a deterministic function of their inputs. More generally, if some of the NAND gates were to act erratically, and if the input sources to the circuit also acted in a random fashion, then such a circuit could also be modelled by a CB

net. In this non-deterministic CB net, the signal flowing in each wire would have a certain probability of being 0 and of being 1. If the signal in a certain wire were measured so that one knew that it was definitely 1, then one would want to revise the probability distributions for the signals in all other wires so as to reflect the new evidence. Hence, conditional probabilities and Bayes's rule arise naturally when considering probabilistic nets.

As we shall see, a major difference between CB nets and QB nets is that for QB nets, one assigns complex amplitudes rather than probabilities to each node.

2. THEORY OF CB NETS

In this section, we will review the simple theory of CB nets. The next section will present some examples.

We call a *graph* (or a diagram or an architecture) a collection of nodes with arrows connecting some pairs of these nodes. The arrows of the graph must satisfy certain constraints that will be specified below. We call a *labelled graph* a graph whose nodes are labelled. A *CB net* consists of two parts: a labelled graph with each node labelled by a random variable, and a collection of node matrices, one matrix for each node. These two parts must satisfy certain constraints that will be specified below.

We define two kinds of arrows: *internal arrows* are those that have a starting node and a different ending one; *external arrows* are those that have a starting node but no ending one. We define two types of nodes: *external nodes* are those that have a single external arrow leaving it, and *internal nodes* are those that have one or more internal arrows leaving it. It is also common to use the terms *root node* or *prior probability node* for a node which has no incoming arrows, only outgoing ones. We restrict all nodes of a graph to be either internal or external. Hence, no nodes may have both an external and one or more internal arrows leaving it.

We define each node of a CB net to represent a numerical value, and the whole net to represent the product of all these node values. We assign a numerical value to each node as follows. First, we assign a random variable to each node. Suppose the random variables assigned to the N nodes are $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$. Define $Z_N = \{1, 2, \cdots, N\}$. For any finite set S, let $|S|$ be the number of elements in S. If $S = \{k_1, k_2, \cdots, k_{|S|}\} \subset Z_N$, define $(x.)_S = (x_{k_1}, x_{k_2}, \cdots, x_{k_{|S|}})$ and $(\underline{x}.)_S = (\underline{x}_{k_1}, \underline{x}_{k_2}, \cdots, \underline{x}_{k_{|S|}})$. Sometimes, we also abbreviate $(x.)_{Z_N}$ (i.e., the vector that includes all the possible x_j components) by just $x.$, and $(\underline{x}.)_{Z_N}$ by just $\underline{x}.$. For $j \in Z_N$, we imagine node $\underline{x}_j$ to lie in state x_j (See Fig.1). We also imagine all arrows leaving the node $\underline{x}_j$ to lie in state x_j, and thus we label all of them x_j. At this point we've shown how to label each arrow in the graph by x_k for some $k \in Z_N$. Define S_j to be the set of all k such that an arrow labelled x_k enters node $\underline{x}_j$. Now we assign a value $P[x_j|(x.)_{S_j}]$ to node $\underline{x}_j$. $P[x_j|(x.)_{S_j}]$ is what we referred to earlier as a *node matrix*; x_j is the matrix's *row index* and $(x.)_{S_j}$ is its *column index*. As the notation

suggests, we assume that the values $P[x_j|(x.)_{S_j}]$ are conditional probabilities; i.e., that they satisfy

$$P[x_j|(x.)_{S_j}] \geq 0 \ , \tag{2.1}$$

$$\sum_{x_j} P[x_j|(x.)_{S_j}] = 1 \ , \tag{2.2}$$

where the sum in Eq.(2.2) is over all states x_j that the random variable $\underline{x}_j$ can assume, and where Eqs. (2.1) and (2.2) must be satisfied for all $j \in Z_N$ and for all possible values of the vector $(\underline{x}.)_{S_j}$ of random variables. The left hand side of Eq.(2.2) is just the sum over the entries along a column of a node matrix. The CB net is taken to represent the product of all the probabilities $P[x_j|(x.)_{S_j}]$ for $j \in Z_N$. This product is a function $P(x.)$ of the current states $x_1, x_2, \cdots, x_N$ of the nodes. Thus,

$$P(x.) = \prod_{j \in Z_N} P[x_j|(x.)_{S_j}] \ . \tag{2.3}$$

We require that

$$\sum_{x.} P(x.) = 1 \ , \tag{2.4}$$

as expected for a joint probability distribution of the random variables $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$.

Next, to illustrate the CB net concepts just presented, we will discuss all possible CB nets with two or three nodes.

For 2 nodes labelled by random variables $\underline{x}$ and $\underline{y}$, there are only 2 possible labelled graphs, depending on whether the arrow points from $\underline{x}$ to $\underline{y}$ (Fig.2a) or vice versa (Fig.2b).

Figure 2a is diagrammatic notation for the following equation:

$$P(x, y) = P(y|x)P(x) \ . \tag{2.5}$$

Notice that each probability factor on the right hand side of Eq.(2.5) is represented by a node in Fig.2a. Notice also that the prior probability $P(x)$, since it has no conditions placed on it, is portrayed in Fig.2a by a node which has no arrows pointing into it. The probability $P(y|x)$, on the other hand, depends on the value x of node $\underline{x}$, and thus it is represented by a node with an arrow labelled x pointing into it. According to Eq.(1.1), Eq.(2.5) is a *tautology* (i.e., a statement that is always true). Thus, the net of Fig.2a may represent any probability distribution of two random variables $\underline{x}$ and $\underline{y}$.

Figure 2b is diagrammatic notation for the following equation:

$$P(x, y) = P(x|y)P(y) \ . \tag{2.6}$$

Again, this equation is a tautology. Thus, like the net of Figs.2a, the net of Fig.2b may represent an arbitrary probability distribution of two random variables $\underline{x}$ and $\underline{y}$. (A collorary is that two CB nets with different labelled graphs may still represent the same probability distribution.)

Figure 3a is diagrammatic notation for

$$P(x,y,z) = P(z|x,y)P(y)P(x)\ . \tag{2.7}$$

Contrary to Eqs.(2.5) and (2.6), Eq.(2.7) does not represent a tautology: not all probability distributions of three random variables $\underline{x}$, $\underline{y}$ and $\underline{z}$ must satisfy Eq.(2.7). In fact, summing both sides of the last equation over the values of $\underline{z}$ yields

$$P(x,y) = P(y)P(x)\ , \tag{2.8}$$

i.e., $\underline{x}$ and $\underline{y}$ are *independent*. Thus, the net of Fig.3a represents two independent random variables $\underline{x}$ and $\underline{y}$.

Figure 3b is diagrammatic notation for the following equation:

$$P(x,y,z) = P(z|x)P(y|x)P(x)\ . \tag{2.9}$$

Dividing both sides of Eq.(2.9) by $P(x)$ yields:

$$P(y,z|x) = P(z|x)P(y|x)\ . \tag{2.10}$$

Even though $\underline{y}$ and $\underline{z}$ are not necessarily independent, they are *conditionally independent*, i.e., they are independent at fixed $\underline{x} = x$. Thus, the net of Fig.3b represents two conditionally independent random variables $\underline{y}$ and $\underline{z}$.

Figure 3c is diagrammatic notation for the equation:

$$P(x,y,z) = P(z|y)P(y|x)P(x)\ . \tag{2.11}$$

Random variables $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$ form an N *step Markov chain* (N may be infinite) if $P(x_{n+1}|x_n, x_{n-1}, \cdots, x_1) = P(x_{n+1}|x_n)$ for $n = 1, 2, \cdots, N-1$, i.e., if the (n+1)th step x_{n+1} depends only on the step x_n immediately preceding it. Clearly, the random variables $\underline{x}$, $\underline{y}$ and $\underline{z}$ of Fig.3c represent a 3 step Markov chain.

Figure 3d is diagrammatic notation for the equation:

$$P(x,y,z) = P(z|y,x)P(y|x)P(x)\ . \tag{2.12}$$

Replacing the conditional probabilities on the right hand side of Eq.(2.12) by their definitions in terms of probabilities without conditions, we obtain $P(x,y,z)$ for the right hand side. Thus, Eq.(2.12) is a tautology, and the net of Fig.3d can represent any probability distribution of three random variables $\underline{x}$, $\underline{y}$ and $\underline{z}$.

The graph of Fig.3d is *acyclic*. That is, it does not contain any cycles (a *cycle* is a closed path of arrows with the arrows all pointing in the same sense). On the other hand, the graph of Fig.4 is *paracyclic* (i.e., it contains at least one cycle). In

fact, the whole graph of Fig.4 is a cycle. The net of Fig.4 shall be forbidden because, if one sums over its free indices (x, y, z), one does not always obtain unity, as must be the case if the net is to represent a probability distribution $P(x, y, z)$. For example, if we assume that $\underline{x}, \underline{y}$ and $\underline{z}$ can assume only states 0 and 1, and if $P(y|x) = \delta_{y,x}$, $P(x|z) = \delta_{x,z}$ and $P(z|y) = \delta_{z,y}$, where $\delta_{x,y}$ is the Kronecker delta function, then Fig.4 adds up to $\sum_{x,y,z} \delta_{x,z}\delta_{z,y}\delta_{y,x} = 2$. The acyclic net of Fig.3d does add up to one, as proven in diagrammatic notation in Fig.5. In this figure, summation over the states of an arrow is indicated by giving the arrow a double shaft. Note that in Fig.5, we first add over the index z of the external arrow. This produces a new external arrow, and we add over its index y, and so on. Each time we add over the index of the current external arrow until finally we get unity. One cannot follow this summation procedure to show that the net of Fig.4 adds up to one. Indeed, Fig.4 has no external arrow to begin the procedure with.

Certain aspects of the preceding discussion of nets with two or three nodes can be generalized to any number N of nodes.

For any number of nodes, we will say that a graph is *fully connected* if it is acyclic, and if everyone of its nodes is connected to all other nodes by either incoming or outgoing arrows. We will say that a net is fully connected if its graph is fully connected. Any fully connected CB net represents a completely general joint probability distribution of the random variables labelling its nodes. Indeed, it is always possible to label the nodes of an N node fully connected CB net so that the net represents the right hand side of the following equation:

$$P(x.) = P(x_N|x_{N-1}, x_{N-2}, \cdots, x_1)P(x_{N-1}|x_{N-2}, x_{N-3}\cdots, x_1)\cdots P(x_2|x_1)P(x_1)\ . \tag{2.13}$$

And this last equation is a tautology. To label the nodes of a fully connected net so that Eq.(2.13) applies to it, one proceeds as follows. There always exists exactly one external node. This is why. The external node, call it x_N, must exist. Otherwise, all nodes would have at least one outgoing internal arrow. Then one could start from any node and travel along one of its outgoing internal arrows to reach another node, and so on, until one came back to a previously visited node. Therefore, the graph would not be acyclic. The external node x_N is unique because, since all other nodes have outgoing arrows that point to x_N, all other nodes must be internal. Remove x_N from the graph. For the same reasons as before, the resulting diminished graph contains a unique external node. Call the latter node x_{N-1}. Continue removing nodes in this way until all the nodes are labelled $x_1, x_2, \cdots, x_N$. Since Eq.(2.13) suggests that x_j occurs after x_{j-1} for $j = 2, 3, \cdots, N$, we call this node labelling (ordering) the *chronological labelling* of the graph. Two colloraries of the preceding proof are that fully connected graphs have a single external node, and that all fully connected N-node labelled graphs are identical once they are relabelled in the fashion described above. Figure 6a shows the fully connected four node graph with its chronological labelling. By deforming Fig.6a into a topologically equivalent diagram, one obtains

Fig.6b, a more "stylized" version of the same thing. Fig.6b might make more clear to some readers how the arrows of a fully-connected graph are organized.

Note that one can relabel any graph chronologically, even if it isn't fully connected. Indeed, given any graph G, one may add arrows to it to form a fully connected graph $\overline{G}$. Call $\overline{G}$ a *completion* of G. Sometimes it is possible to add arrows to G in two different ways and arrive at two different completions. Hence completions are not always unique. Any completion of a graph G may be labelled chronologically following the procedure described above. This, in turn, gives the graph G a chronological ordering, albeit, not a necessarily unique one. Suppose that $\underline{x}$ and $\underline{y}$ are two nodes in a graph G. We say that $\underline{x}$ *precedes* $\underline{y}$ and write $\underline{x} < \underline{y}$ if, for any completion of G, there exist integers j and j' with $j < j'$ so that $\underline{x} = \underline{x}_j$ and $\underline{y} = \underline{x}_{j'}$. We say that $\underline{x}$ and $\underline{y}$ are *concurrent* and write $\underline{x} \sim \underline{y}$ if $\underline{x}$ precedes $\underline{y}$ in some completions but $\underline{y}$ precedes $\underline{x}$ in others. And we say that $\underline{x}$ *succeeds* or *follows* $\underline{y}$ and write $\underline{x} > \underline{y}$ if $\underline{y} < \underline{x}$.

Call a *CB pre-net* a labelled graph and an accompanying set of node matrices that satisfy Eqs.(2.1), (2.2) and (2.3), but don't necessarily satisfy the overall normalization condition Eq.(2.4). An acyclic CB pre-net always satisfies Eq.(2.4), and a paracyclic CB pre-net may not satisfy Eq.(2.4). This is why. Following the procedure just discussed, the nodes of any acyclic pre-net can be relabelled chronologically as $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$. The relabelled graph, even if it is not fully connected, corresponds to the right hand side of Eq.(2.13), except that some of the conditional probabilities on the right hand side of Eq.(2.13) might include redundant conditioning. (If a conditional probability $P(x|y,R)$ is known to satisfy $P(x|y,R) = P(x|y)$, then we would say that the expression $P(x|y,R)$ shows redundant conditioning on R.) This correspondence between any acyclic pre-net and the right hand side of Eq.(2.13) guarantees that acyclic pre-nets will satisfy the overall normalization condition Eq.(2.4). On the other hand, as we've shown with the 3-node pre-net of Fig.4, an N-node paracyclic pre-net may not reduce to unity upon summing over its free indices. If one tries the process of "adding over all the current external arrows" on a pre-net with a cycle embedded in it, at some point the process comes to a stop and cannot be completed due to a lack of a current external arrow to sum over next. Note that, in some sense, Eq.(2.13) embodies the *principle of causality*. Thus, acyclic CB pre-nets preserve causality, and paracyclic CB pre-nets violate this principle. If one considers only acyclic graphs, as we shall do henceforth, then there is no difference between CB nets and CB pre-nets.

Note that if one sums both sides of Eq.(2.11) over y and divides by $P(x)$, one obtains

$$P(z|x) = \sum_y P(z|y)P(y|x) \ . \tag{2.14}$$

This last equation, valid for a Markov chain $(\underline{x}, \underline{y}, \underline{z})$, is the so called Chapman-Kolgomorov equation. Equation (2.14) is represented diagramatically in Fig.7. If one sums both sides of Eq.(2.12) over y and divides by $P(x)$, one obtains

$$P(z|x) = \sum_y P(z|y,x)P(y|x)\ . \tag{2.15}$$

The last equation is a generalization of the Chapman-Kolgomorov equation to arbitrary random variables $\underline{x}, \underline{y}, \underline{z}$ that don't necessarily form a Markov chain. Equation (2.15) is represented diagramatically in Fig.8. Note that in both Figs.7 and 8, we follow a process of adding over some of the arrows of a net to obtain a new net with fewer nodes. This process may be called *coarsening* or *data compression*, because it reduces the number of nodes and because the joint probability distribution of the new net carries less information than the joint probability distribution of the old one. In general, if we start with N nodes $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$, and we sum over $x_1, x_2, \cdots, x_k$, then the resulting probability distribution of the variables $x_{k+1}, x_{k+2}, \cdots, x_N$ can always be represented by a fully connected net with nodes $\underline{x}_{k+1}, \underline{x}_{k+2}, \cdots, \underline{x}_N$.

In modelling a classical physical situation by a CB net, if one knows very little about the nature of the probability distributions involved, one can always use a fully connected net. Later on, if one learns that certain pairs of random variables are conditionally independent, one may remove the arrows connecting these pairs of variables without changing the value of the full net.

Once we have designed a net architecture and the net's node matrices have been calculated, what next? How to use this information? The information is sufficient for calculating the joint probability $P(x_1, x_2, \cdots, x_N)$, where $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$ are the nodes of the net. From this joint probability, one can calculate $P(x_a|x_b, x_c, \cdots)$, the probability that any one node $\underline{x}_a$ of the net assumes one of its states x_a, conditioned on several other nodes $\underline{x}_b, \underline{x}_c, \cdots$ of the net assuming respective states $x_b, x_c, \cdots$. To go from $P(x_1, x_2, \cdots, x_N)$ to $P(x_a|x_b, x_c, \cdots)$, one can sum $P(x_1, x_2, \cdots, x_N)$ over those variables which are not in the list $x_a, x_b, x_c, \cdots$ to get $P(x_a, x_b, x_c, \cdots)$. One can obtain $P(x_b, x_c, \cdots)$ similarly, and then divide the two. However, this brute force method takes no advantage of the particular topology of the net, and so it is very labor intensive. Artificial intelligence and expert systems workers have found an algorithm for calculating $P(x_a|x_b, x_c, \cdots)$ from $P(x_1, x_2, \cdots, x_N)$ that takes advantage of the net topology to reduce dramatically the number of arithmetical operations required. For a discussion of this great achievement, the reader can consult Refs.[1]-[8]. The software package of Ref.[9] implements this fast algorithm.

3. EXAMPLES OF CB NETS

The references at the end of this paper present examples of CB nets used in medical diagnosis[4],[6], monitoring of processes[1], genetics[5], etc. And this is just a small fraction of the possible applications of CB nets. Indeed, any probabilistic model can be discussed in terms of CB nets. In this section, we do not intend to present a comprehensive collection of CB net examples, but only to give a few simple introductory ones.

(a)DIGITAL CIRCUITS

Figure 9 shows a Bayesian net version of an AND gate. The random variables $\underline{x}, \underline{y}$ and $\underline{z}$ are *binary* (i.e. they assume values in $\{0, 1\}$.) The node matrices for this net consist of the prior probabilities $P(x), P(y)$ and of the conditional probability $P(z|x, y)$. $P(z|x, y)$ is given by the table in Fig.9. With states 0 and 1 standing for false and true, respectively, this table is what one would expect if $z = (\ x$ and $y\ )$. Note that the columns of the node matrix $P(z|x, y)$ add up to one, as required by Eq.(2.2). A node matrix whose entries are all either 0 or 1 (like $P(z|x, y)$ in this example) will be called a *deterministic node matrix*.

CB nets representing OR and NOT gates can be defined analogously to the AND net. Then one can construct CB nets for any combination of AND, OR and NOT gates.

(b)CONSTRAINT NODES

Figure 10 shows a net for the binary random variables $\underline{x}, \underline{y}$ and the random variable $\underline{z} \in \{0, 1, 2\}$. To fully specify this net, one must give the prior probabilities $P(x), P(y)$ and the conditional probability $P(z|x, y)$. The node matrix $P(z|x, y)$ is given by the table of Fig.10. This node matrix is deterministic and agrees with $z = x + y$. Note that if we know for sure that $x + y = 1$, then we can "fix" the *sum node* $\underline{z}$ to one, and calculate probabilities, like $P(x|\underline{z} = 1)$, which have $\underline{z} = 1$ as evidence. In such a case, we call $\underline{z}$ a *constraint node*, because it is used to enforce the constraint $x + y = 1$.

As another example of a net that possesses a node which is useful to constrain, consider Fig.11. In this net the random variables $\underline{x}, \underline{y}$ and $\underline{z}$ are binary. To fully specify the net, one must give prior probabilities $P(x)$, $P(y)$ and the conditional probability $P(z|x, y)$. $P(z|x, y)$ is given by the table in Fig.11. This table is what one would expect if $z =$ (if x then y). Since an "if x then y" statement does not say anything about what to do when $x = 0$, the probabilities $P(z|x, y)$ are arbitrary and not necessarily deterministic when $x = 0$. The net of Fig.11 could be used by fixing the *if-then node* $\underline{z}$ to one and considering only probabilities with $\underline{z} = 1$ as evidence. In such a case, we would say that the $\underline{z}$ node was a constraint node.

(c)CLAUSER-HORNE EXPERIMENT

Consider the Clauser-Horne experiment, which is used to observe violations of Bell-type inequalities[12]. Two particles, call them 1 and 2, are created at a common vertex and fly apart. Let λ represent the "hidden variables". Particle 1 is subjected to a spin measurement along the direction A with outcome $x_1^A \in \{+, -\}$ or along the direction A' with outcome $x_1^{A'} \in \{+, -\}$. Particle 2 is subjected to a spin measurement along the direction B with outcome $x_2^B \in \{+, -\}$ or along the direction B' with outcome $x_2^{B'} \in \{+, -\}$. One can draw the net of Fig.12, which has nodes $\underline{\lambda}, \underline{x}_1^{\theta_1}$ and $\underline{x}_2^{\theta_2}$. (Figure 12 really represents 4 nets, one for each of the following possibilities: $(\theta_1, \theta_2) = (A, B), (A, B'), (A', B), (A', B')$). To specify the net, one must give probabilities $P(\lambda)$, $P(x_1^{\theta_1}|\lambda)$ and $P(x_2^{\theta_2}|\lambda)$. Note that if one sums the net of Fig.12 over λ, one gets

$$P(x_1^{\theta_1}, x_2^{\theta_2}) = \sum_\lambda P(x_1^{\theta_1}|\lambda) P(x_2^{\theta_2}|\lambda) P(\lambda) \ , \tag{3.1}$$

which is the starting point in the derivation of the Bell inequalities for the Clauser-Horne experiment.

As a slightly more complicated experiment, one might choose at random whether $\theta_1 = A$ or A' and whether $\theta_2 = B$ or B'. Such an experiment can be represented by a CB net with nodes $\underline{\theta}_1, \underline{x}_1, \underline{\theta}_2, \underline{x}_2$ and $\underline{\lambda}$, where $\underline{x}_j$ for $j \in \{1, 2\}$ represents the outcome of a measurement on particle j. See Fig.13. This net is specified if we give probabilities $P(x_1|\theta_1, \lambda), P(\theta_1), P(x_2|\theta_2, \lambda), P(\theta_2)$ and $P(\lambda)$. Summing the net over λ and dividing by $P(\theta_1)P(\theta_2)$ yields an equation analogous to Eq.(3.1):

$$P(x_1, x_2|\theta_1, \theta_2) = \sum_\lambda P(x_1|\theta_1, \lambda) P(x_2|\theta_2, \lambda) P(\lambda) \ . \tag{3.2}$$

(d)RANDOM WALK

Suppose that a particle moves in a straight line, taking unit length steps either forwards or backwards, with probabilities p_+ and p_-, respectively, where $p_+ + p_- = 1$. Let x_j be the position of the particle at time j, with $j \in \{0, 1, 2, \cdots, \}$. Assume x_0, the starting position, is zero. Define Δx_j by $\Delta x_j = x_j - x_{j-1}$, for $j \in \{1, 2, \cdots, \}$. Figure 14a shows a CB net that represents the probability distribution of $\underline{x}_0, \underline{x}_1, \cdots$ and $\underline{\Delta x}_1, \underline{\Delta x}_2, \cdots$. Clearly, $x_j \in \{0, \pm 1, \pm 2, \cdots, \pm j\}$ and $\Delta x_j \in \{\pm 1\}$. To fully specify the net of Fig.14a, we must give the probability matrices associated with each node. These matrices are

$$\begin{array}{l} P(\underline{x}_0 = 0) = 1 \ , \\ P(\underline{\Delta x}_j \pm 1) = p_\pm \ \text{ for } j = 1, 2, \cdots \ , \\ P(\underline{x}_j = y|\underline{x}_{j-1} = x, \underline{\Delta x}_j \pm 1) = \delta(y, x \pm 1) \ \text{ for } j = 1, 2, \cdots \ , \end{array} \tag{3.3}$$

where δ is the Kronecker delta function. By summing the net of Fig.14a over all possible values of the indices Δx_j that label the arrows from the $\underline{\Delta x}_j$ to the $\underline{x}_j$ nodes, one obtains Fig.14b. For this coarser net, one obtains

$$\begin{array}{l} P(\underline{x}_0 = 0) = 1 \ , \\ P(\underline{x}_j = y|\underline{x}_{j-1} = x) = p_+ \delta(y, x+1) + p_- \delta(y, x-1) \ \text{ for } j = 1, 2, \cdots \ . \end{array} \tag{3.4}$$

One may go even further: For any $k \in \{0, 1, 2 \cdots\}$, one may sum Fig.14b over all x_j such that $j \notin \{0, k\}$. One obtains the net of Fig.14c, with

$$P(\underline{x}_0 = 0) = 1 \ , \tag{3.5a}$$

$$P(\underline{x}_k = x_k|\underline{x}_0 = 0) = \left(\begin{array}{c} r+s \\ r \end{array} \right) p_+^r p_-^s \ , \tag{3.5b}$$

where $r-s=x_k$ and $r+s=k$, and the first factor on the right hand side of Eq.(3.5b) is a combinatorial factor.

4. THEORY OF QB NETS

In this section, we will define QB nets and explain how to use them to calculate quantum mechanical conditional probabilities. The next section will give examples of QB nets.

Like a CB net, a *QB net* consists of two parts: a labelled graph and a collection of node matrices. These two parts must satisfy certain constraints that will be specified below.

External and internal arrows, external and internal nodes, and root nodes are all defined in the same way for QB nets as for CB nets. All nodes of a QB net must be either internal or external.

We define each node of a QB net to represent a numerical value, and the whole net to represent the product of all these node values. We assign a numerical value to each node as follows. First, we assign a random variable to each node. Suppose the random variables assigned to the N nodes are $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_N$. Define $Z_N = \{1, 2, \cdots, N\}$. For any finite set S, let $|S|$ be the number of elements in S. If $S = \{k_1, k_2, \cdots, k_{|S|}\} \subset Z_N$, define $(x.)_S = (x_{k_1}, x_{k_2}, \cdots, x_{k_{|S|}})$ and $(\underline{x}.)_S = (\underline{x}_{k_1}, \underline{x}_{k_2}, \cdots, \underline{x}_{k_{|S|}})$. Sometimes, we also abbreviate $(x.)_{Z_N}$ (i.e., the vector that includes all the possible x_j components) by just $x.$, and $(\underline{x}.)_{Z_N}$ by just $\underline{x}.$. For $j \in Z_N$, we imagine node $\underline{x}_j$ to lie in state x_j. We also imagine all arrows leaving the node $\underline{x}_j$ to lie in state x_j, and thus we label all of them x_j. At this point we've shown how to label each arrow in the graph by x_k for some $k \in Z_N$. Define S_j to be the set of all k such that an arrow labelled x_k enters node $\underline{x}_j$. Now we assign a complex number $A[x_j|(x.)_{S_j}]$ to node $\underline{x}_j$. $A[x_j|(x.)_{S_j}]$ is the *node matrix* for node $\underline{x}_j$; x_j is the matrix's *row index* and $(x.)_{S_j}$ is its *column index*. We require that the quantities $A[x_j|(x.)_{S_j}]$ be probability amplitudes that satisfy

$$\sum_{x_j} \left| A[x_j|(x.)_{S_j}] \right|^2 = 1 \,, \tag{4.1}$$

where the sum in Eq.(4.1) is over all states x_j that the random variable $\underline{x}_j$ can assume, and where Eq.(4.1) must be satisfied for all $j \in Z_N$ and for all possible values of the vector $(\underline{x}.)_{S_j}$ of random variables. The QB net is taken to represent the product of all the probability amplitudes $A[x_j|(x.)_{S_j}]$ for $j \in Z_N$. This product is a function $A(x.)$ of the current states $x_1, x_2, \cdots, x_N$ of the nodes. Thus,

$$A(x.) = \prod_{j \in Z_N} A[x_j|(x.)_{S_j}] \,. \tag{4.2}$$

Let Z_N^{ext} be the set of all $j \in Z_N$ such that $\underline{x}_j$ is an external node, and let Z_N^{int} be

the set of those $j \in Z_N$ such that $\underline{x}_j$ is an internal node. Clearly, Z_N^{ext} and Z_N^{int} are disjoint and their union is Z_N. We require $A(x.)$ to satisfy

$$\sum_{(x.)_{Z_N^{ext}}} \left| \sum_{(x.)_{Z_N^{int}}} A(x.) \right|^2 = 1 \tag{4.3}$$

and

$$\sum_{x.} |A(x.)|^2 = 1 \; . \tag{4.4}$$

Note that as a consequence of Eqs.(4.1) and (4.4), given any QB net, one can construct a special CB net by replacing the value $A[x_j|(x.)_{S_j}]$ of each node by its magnitude squared. We call this special CB net the *parent CB net* of the QB net from which it was constructed. We call it so because, given a parent CB net, one can replace the value of each node by its square root times a phase factor. For a different choice of phase factors, one generates a different QB net. Thus, a parent CB net may be used to generate a whole family of QB nets.

A *QB pre-net* is a labelled graph and an accompanying set of node matrices that satisfy Eqs.(4.1), (4.2) and (4.3), but don't necessarily satisfy Eq.(4.4). A QB pre-net that is acyclic satisfies Eq.(4.4), because its parent CB pre-net is acyclic and this implies that Eq.(4.4) is satisfied. If one considers only acyclic graphs, as we shall do henceforth, then there is no difference between QB nets and QB pre-nets.

In the second quantized formulation of quantum mechanics, one speaks of M *modes* represented by M *annihilation operators* $a_1, a_2, \cdots, a_M$, which obey certain commutation relations amongst themselves. Define $Z_{0+} = \{0, 1, 2, \cdots\}$ and let $n_i \in Z_{0+}$ for $i = 1$ to M. For M modes, one uses quantum states like

$$\frac{a_1^{\dagger n_1}}{\sqrt{n_1!}} \frac{a_2^{\dagger n_2}}{\sqrt{n_2!}} \cdots \frac{a_M^{\dagger n_M}}{\sqrt{n_M!}} |0\rangle \; . \tag{4.5}$$

The state given by Eq.(4.5) is specified by a vector $(n_1, n_2, \cdots, n_M)$ of *occupation numbers*. For the rest of this paper, we will consider only QB nets whose states x_j are vectors $(n_{j,1}, n_{j,2}, \cdots, n_{j,K_j})$ of occupation numbers. Define Γ to be the set of all α such that n_α is an occupation number of the QB net under consideration. For example, $\alpha = (j, 1)$ in $n_{j,1}$, where $n_{j,1}$ is a component of $x_j = (n_{j,1}, n_{j,2}, \cdots, n_{j,K_j})$. For any set $\Gamma' = \{\alpha_1, \alpha_2, \cdots, \alpha_{|\Gamma'|}\} \subset \Gamma$, let $(n.)_{\Gamma'} = (n_{\alpha_1}, n_{\alpha_2}, \cdots, n_{\alpha_{|\Gamma'|}})$ and $(\underline{n}.)_{\Gamma'} = (\underline{n}_{\alpha_1}, \underline{n}_{\alpha_2}, \cdots, \underline{n}_{\alpha_{|\Gamma'|}})$. We will sometimes abbreviate $(n.)_\Gamma$ by $n.$ and $(\underline{n}.)_\Gamma$ by $\underline{n}.$. For each j, define Γ_j to be the set such that $(n.)_{\Gamma_j}$ is the occupation number vector specifying the state of node $\underline{x}_j$. Thus, $x_j = (n.)_{\Gamma_j}$ and $\underline{x}_j = (\underline{n}.)_{\Gamma_j}$. If $\underline{x}_j$ is an external node, call all the components of $(n.)_{\Gamma_j}$ *external occupation numbers*, and if $\underline{x}_j$ is an internal node, call all the components of $(n.)_{\Gamma_j}$ *internal occupation numbers*. Let Γ^{ext} be the set of all α such that n_α is an external occupation number, and let Γ^{int} be the

set of all α such that n_α is an internal occupation number. Clearly, Γ^{int} and Γ^{ext} are disjoint and their union is Γ. Note that $(x.)_{Z_N^{ext}} = (n.)_{\Gamma^{ext}}$, $(x.)_{Z_N^{int}} = (n.)_{\Gamma^{int}}$, and $(x.)_{Z_N} = (n.)_\Gamma$ (Analogous statements with x and n underlined to indicate random variables also hold).

Consider a classical probability distribution $P(n.)$. For any $\Gamma_0 \subset \Gamma$, we may define the characteristic function χ_c by

$$\chi_c[(n.)_{\Gamma_0}] = \sum_{m.} P(m.) \prod_{\alpha \in \Gamma_0} \delta(m_\alpha, n_\alpha) \ , \tag{4.6}$$

where δ is the Kronecker delta function. (The "c" subscript in χ_c stands for "classical".) If Γ_H and Γ_E are disjoint subsets of Γ, one defines the classical conditional probability that $(\underline{n}.)_{\Gamma_H} = (n.)_{\Gamma_H}$ (hypothesis) given that $(\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}$ (evidence), by

$$P\left[(\underline{n}.)_{\Gamma_H} = (n.)_{\Gamma_H} | (\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}\right] \ = \ \frac{\chi_c[(n.)_{\Gamma_H}, (n'.)_{\Gamma_E}]}{\chi_c[(n'.)_{\Gamma_E}]} \ . \tag{4.7}$$

By setting $\Gamma_E = \phi$ in the last equation, we conclude that for any $\Gamma_0 \subset \Gamma$,

$$P\left[(\underline{n}.)_{\Gamma_0} = (n.)_{\Gamma_0}\right] \ = \ \chi_c[(n.)_{\Gamma_0}] \ . \tag{4.8}$$

According to the definition Eq.(4.6), if Γ_0 and Γ_0' are disjoint subsets of Γ, then

$$\sum_{(n'.)_{\Gamma_0'}} \chi_c[(n.)_{\Gamma_0}, (n'.)_{\Gamma_0'}] = \chi_c[(n.)_{\Gamma_0}] \ . \tag{4.9}$$

Equations (4.8) and (4.9) imply that

$$\sum_{(n'.)_{\Gamma_0'}} P[(n.)_{\Gamma_0}, (n'.)_{\Gamma_0'}] = P[(n.)_{\Gamma_0}] \ , \tag{4.10}$$

for $\Gamma_0, \Gamma_0' \subset \Gamma$, $\Gamma_0 \cap \Gamma_0' = \phi$. Equation (4.9), when applied to Eq.(4.7), yields

$$\sum_{(n.)_{\Gamma_H}} P[(n.)_{\Gamma_H} | (n'.)_{\Gamma_E}] = 1 \ . \tag{4.11}$$

Furthermore, from Eq.(4.7) it is obvious that

$$P[(n.)_{\Gamma_H} | (n'.)_{\Gamma_E}] \geq 0 \ . \tag{4.12}$$

Equations (4.11) and (4.12) imply that Eq.(4.7) is an adequate definition of conditional probability.

Now consider a quantum mechanical probability amplitude $A(n.)$. For any $\Gamma_0 \subset \Gamma$, we define the characteristic function χ by

$$\chi[(n.)_{\Gamma_0}] = \sum_{(m.)_{\Gamma^{ext}}} \left| \sum_{(m.)_{\Gamma^{int}}} A(m.) \prod_{\alpha \in \Gamma_0} \delta(m_\alpha, n_\alpha) \right|^2 . \tag{4.13}$$

If Γ_H and Γ_E are disjoint subsets of Γ, one defines the quantum mechanical conditional probability that $(\underline{n}.)_{\Gamma_H} = (n.)_{\Gamma_H}$ (hypothesis) given that $(\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}$ (evidence), by

$$P\left[(\underline{n}.)_{\Gamma_H} = (n.)_{\Gamma_H} | (\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}\right] \;=\; \frac{\chi[(n.)_{\Gamma_H}, (n'.)_{\Gamma_E}]}{\sum_{(m.)_{\Gamma_H}} \chi[(m.)_{\Gamma_H}, (n'.)_{\Gamma_E}]} . \tag{4.14}$$

Note that the denominator of the right hand side of Eq.(4.14) depends on both Γ_E and Γ_H, unlike the analogous denominator in the classical definition Eq.(4.7). In quantum mechanics, with χ_c replaced by χ, Eq.(4.8) is not true for all $\Gamma_0 \subset \Gamma$ (although it is true if $\Gamma_0 \subset \Gamma^{ext}$). In quantum mechanics, with χ_c replaced by χ, Eqs.(4.9) and (4.10) are not necessarily true, whereas Eqs.(4.11) and (4.12) are obviously always true. Because Eqs.(4.11) and (4.12) are satisfied in quantum mechanics, Eq.(4.14) is an adequate definition of conditional probability.

Note that both the classical and quantum mechanical definitions of conditional probability Eqs. (4.7) and (4.14) implicitly assume that we perform non-destructive measurements on any internal nodes that might be measured. Hence, if a particle is detected at an internal node, it is allowed to continue past that node so that it can reach other nodes further downstream. If one wishes to perform a destructive measurement on a particle when it passes through an internal node $\underline{x}$, then one is representing the physical situation by the wrong CB or QB net; what is required is a net that has the node $\underline{\mathrm{x}}$ as an external node.

In Appendix A, the classical and quantum mechanical definitions of conditional probability Eqs.(4.7) and (4.14) are generalized so that they allow either sharp or fuzzy hypotheses and pieces of evidence. If the evidence narrows the set of possible values for $\underline{n}_1$ to a single number, say $\underline{n_1} = 1$, then we say that the evidence for $\underline{n}_1$ is *sharp*. If the evidence doesn't do this, then we say that the evidence for $\underline{n}_1$ is *fuzzy*. Sharp and fuzzy hypotheses are defined analogously.

Note that if we had used Eq.(4.7) (with χ_c replaced by χ) as the quantum mechanical definition of conditional probability, there would have been no guarantee that Eq.(4.11) would be satisfied. For given Γ_H, Γ_E and $(n'.)_{\Gamma_E}$, define the *quantum non-additivity factor* f_{qna} by

$$f_{qna}[\Gamma_H, \Gamma_E, (n'.)_{\Gamma_E}] = \frac{\sum_{(m.)_{\Gamma_H}} \chi[(m.)_{\Gamma_H}, (n'.)_{\Gamma_E}]}{\chi[(n'.)_{\Gamma_E}]} . \tag{4.15}$$

This quantity will be calculated for the examples in the next section. If $f_{qna} = 1$, then Eq.(4.7) (with χ_c replaced by χ) and Eq.(4.14) agree; if $f_{qna} \neq 1$, then Eq.(4.7) does not give a well defined probability distribution for $(n.)_{\Gamma_H}$ whereas Eq.(4.14) does.

In quantum mechanics, $\chi[(n.)_{\Gamma_H}, (n'.)_{\Gamma_E}]$ is proportional to the number of occurrences of $(\underline{n}.)_{\Gamma_H} = (n.)_{\Gamma_H}$ and $(\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}$ in an experiment that measures all the Γ_H and Γ_E nodes. $\chi[(n'.)_{\Gamma_E}]$ is proportional to the number of occurrences of $(\underline{n}.)_{\Gamma_E} = (n'.)_{\Gamma_E}$ in an experiment that measures only the Γ_E nodes, leaving the Γ_H nodes undisturbed. Thus, Eq.(4.7) (with χ_c replaced by χ) refers to the ratio of the number of occurrences in two different types of experiments (one type measuring the $\Gamma_H \cup \Gamma_E$ nodes and the other only the Γ_E nodes). On the other hand, Eq.(4.14) refers to the ratio of occurrences in a single type of experiment (measuring the $\Gamma_H \cup \Gamma_E$ nodes). It seems to us that the latter ratio is the more useful of the two.

By using CB and QB nets, one is led easily and naturally to calculate probabilities by considering sums over paths, the type of sums advocated by Feynman for quantum mechanics and by Kac for Brownian motion. Indeed, one can express the classical and quantum mechanical definitions of conditional probability Eqs.(4.7) and (4.14) in terms of sums over paths rather than sums over node states. We do so for arbitrary CB and QB nets in Appendix B. Specific examples illustrating the affinity of QB nets with sums over paths can be found in the next section and in Appendix C. Appendix C presents a QB net which yields the Feynman path integral for a single mass particle under the influence of an arbitrary potential. Using an approach similar to that of Appendix C, it should be possible to define QB nets that yield the Feynman integrals employed in non-relativistic and relativistic quantum field theories.

5. EXAMPLES OF QB NETS

In this section, we will present the results of a computer program that calculates conditional probabilities for QB nets. In particular, we shall consider QB nets for experiments containing either two or three Stern-Gerlach magnets. We will restrict our attention to experiments involving a single particle. For simplicity, for each experimental configuration, we will assume that the magnetic field vectors of all the Stern-Gerlach magnets are coplanar.

Figures 15, 16 and 17 show the three kinds of nodes that will be used in this section.

In Fig.15, the triangle represents a root node. This node will stand for $\psi_{n_{z-}n_{z+}}$, with $(n_{z-}, n_{z+}) \in \{(0,1),(1,0)\}$ and $|\psi_{01}|^2 + |\psi_{10}|^2 = 1$. $\psi_{n_{z-}n_{z+}}$ is just the initial wavefunction for the single particle under consideration.

In Fig.16a, the black-filled circle represents a *marginalizer node*. The single incoming arrow is in a state characterized by a vector $(n_1, n_2, \cdots, n_K)$ of occupation numbers. The outgoing arrow is in a state characterized by a single occupation number n'_1. The amplitude associated with the node is

$$A(n'_1|n_1, n_2, \cdots, n_K) = \delta(n'_1, n_1) \ . \tag{5.1a}$$

Thus, a marginalizer node takes a vector of occupation numbers and projects out one

of its components. Note that Eq.(5.1a) satisfies Eq.(4.1).

In Fig.16b, the black-filled circle with a phase factor next to it represents a *phase shifter node*. The single incoming (outgoing) arrow is in a state characterized by a single occupation number n_1 (n_1'). The amplitude associated with the node is

$$A(n_1'|n_1) = e^{i\xi}\delta(n_1', n_1) \ , \tag{5.1b}$$

where ξ is a real constant. Note that Eq.(5.1b) satisfies Eq.(4.1).

In Figs.17, the white-filled circles represent Stern-Gerlach magnets. The outgoing arrows are labelled by a vector $\mathbf{n}_u^\alpha = (n_{u-}^\alpha, n_{u+}^\alpha)$ of occupation numbers. Here the unit vector $\hat{u}$ represents the direction of the magnet's magnetic field, and α labels the magnet ($\hat{u}$ is not enough to label the magnet if the experiment contains more that one magnet whose magnetic field points along the $\hat{u}$ direction.) The vector $\mathbf{n}_u^\alpha \in \{(0,0),(0,1),(1,0)\}$ specifies a state (see Appendix D)

$$(a_{u-}^{\alpha\dagger})^{n_{u-}^\alpha}(a_{u+}^{\alpha\dagger})^{n_{u+}^\alpha}|0\rangle \ . \tag{5.2}$$

The creation operators $a_{u-}^{\alpha\dagger}$ and $a_{u+}^{\alpha\dagger}$ create particles in the $|-_u\rangle$ and $|+_u\rangle$ states respectively. In Fig.17a there is one arrow entering the node, whereas in the Fig.17b there are two. In general, since we are considering a single particle experiment, there may be any number of arrows entering the node, but only one may be in state 1, all others must be in state 0. In Figs.17a and 17b, the amplitude assigned to each node is given by a table next to the graph. Clearly, the tables in Figs.17a and 17b both satisfy Eq.(4.1). For each experimental configuration, we will assume that the magnetic fields of all magnets are coplanar. The plane containing these magnetic field vectors may be chosen to be the X-Z plane with $\phi = 0$. Thus, the matrix elements in the tables of Figs.17a and 17b are given in terms of angular parameters by Eqs.(D.7) and (D.8).

(a) EXPERIMENTS WITH 2 STERN-GERLACH MAGNETS

We will consider 2 configurations with 2 magnets: Fig.18 (the tree diagram) and Fig.19 (the loop diagram). The diagrams are like road maps, with the arrows representing the various roads along which the particle may travel.

A single particle travelling through the experimental configuration of Fig.18 could exit through either the $\underline{n}_{z-}$, $\underline{n}_{u-}$ or $\underline{n}_{u+}$ nodes. There is only one possible path leading to each of these outcomes. Thus, one has

$$\begin{aligned} FI_1 &= A(\pi_1) \ , \\ A(\pi_1) &= \psi_{10} \ , \\ FI_2 &= A(\pi_2) \ , \\ A(\pi_2) &= \langle -_u|+_z\rangle\psi_{01} \ , \\ FI_3 &= A(\pi_3) \ , \\ A(\pi_3) &= \langle +_u|+_z\rangle\psi_{01} \ . \end{aligned} \tag{5.3}$$

For $i \in \{1,2,3\}$, $A(\pi_i)$ is the amplitude for path π_i. For $j \in \{1,2,3\}$, FI_j is a Feynman integral; that is, the sum of the amplitudes for all paths with a given final

state. (See Appendix B). We have already checked that Eq.(4.1) is satisfied. We did so when we defined the wavefunction, marginalizer and Stern-Gerlach nodes. It is easy to show that $\sum_{j=1}^{3} |FI_j|^2 = 1$, so Eq.(4.3) is also satisfied for this net.

A single particle travelling through the experimental configuration of Fig.19 could exit through either the $\underline{n}_{u-}$ or the $\underline{n}_{u+}$ nodes. There are two possible paths leading to each of these two outcomes. Thus, one has

$$\begin{aligned} FI_1 &= A(\pi_1) + A(\pi_2) \ , \\ A(\pi_1) &= \langle -_u | -_z \rangle \psi_{10} \ , \\ A(\pi_2) &= \langle -_u | +_z \rangle \psi_{01} \ , \\ FI_2 &= A(\pi_3) + A(\pi_4) \ , \\ A(\pi_3) &= \langle +_u | -_z \rangle \psi_{10} \ , \\ A(\pi_4) &= \langle +_u | +_z \rangle \psi_{01} \ . \end{aligned} \tag{5.4}$$

As before, for $i \in \{1, 2, 3\}$, $A(\pi_i)$ is the amplitude for path π_i, and for $j \in \{1, 2\}$, FI_j is a Feynman integral. It is easy to show that $\sum_{j=1}^{2} |FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

We will call an arrow or a node *simple* if its state is characterized by a single occupation number (for example, a marginalizer node and its outgoing arrow are both simple). We will say that a net is *fully marginalized* if the incoming arrows of any node that is not a marginalizer are all simple, and all external arrows are simple. The nets of Figs.18 and 19, and, in fact, all the nets considered in this section, are fully marginalized.

For each of the nets of Figs.18 and 19, we used a computer program to calculate probabilities with one or two hypotheses and with zero, one or two pieces of evidence. More precisely, for the 2 QB nets in Figs. 18 and 19, and for their parent CB nets, for $\underline{n}$, $\underline{n}'$, $\underline{m}$ and $\underline{m}' \in \{\underline{n}_{u+}, \underline{n}_{u-}, \underline{n}_{z+}, \underline{n}_{z-}\}$, for all n, n', m and $m' \in \{0, 1\}$, we calculated (using Eqs.(4.7) or (4.14)) the unconditional probabilities $P(n)$ and $P(n, n')$, and the following conditional probabilities:

$$P(\underline{n} = n | \underline{m} = m) \ , \tag{5.5}$$

$$P(\underline{n} = n | \underline{m} = m, \underline{m}' = m') \ , \tag{5.6}$$

$$P(\underline{n} = n, \underline{n}' = n' | \underline{m} = m) \ , \tag{5.7}$$

$$P(\underline{n} = n, \underline{n}' = n' | \underline{m} = m, \underline{m}' = m') \ . \tag{5.8}$$

The table of Fig.20, call it the *evidence-case file*, gives the various sets of evidence that were considered. For example, in case 2 (the row that starts with a 2), we assumed $n_{z+} = 0$, whereas the values of the remaining occupation numbers (n_{z-}, n_{u+}, n_{u-}) were assumed to be unknown (i.e., we assumed there was no evidence

as to whether they were 0 or 1 and this is indicated in Fig.20 with a blank space). In case 11, we assumed $n_{z-} = 1$ and $n_{z+} = 0$, whereas the values of the remaining occupation numbers (n_{u+}, n_{u-}) were assumed to be unknown. Note that for the evidence-cases 2 to 9 we considered 1 piece of evidence (as in Eqs.(5.5) and (5.7)), and for the evidence-cases 10 to 33 we considered 2 pieces of evidence (as in Eqs.(5.6) and (5.8)).

To get numerical values for the probabilities associated with the nets Figs.18 and 19, particular values had to be assumed for the initial wavefunction and for the magnetic field direction of each magnet. We used

$$\psi_{01} = \frac{1+i}{2} \ , \ \ \psi_{10} = \frac{1}{\sqrt{2}} \ , \ \ \theta_u - \theta_z = \frac{\pi}{5} \ . \tag{5.9}$$

For the tree graph of Fig.18, all the probability distributions for the QB net and for its parent CB net were identical. This is not surprising since *tree graphs*, i.e., graphs without any loops, have no interfering paths. That is, for a given final state, there is only one possible path that produces that final state. More interesting results were obtained for the loop graph of Fig.19.

Figures 21 and 22 show our computer program's output for the loop net, for cases 1, 2, 4, 10, 12 (See evidence-case file Fig.20). Figure 21 gives one-hypothesis probabilities of the type Eq.(5.5) and (5.6), whereas Fig.22 gives two-hypotheses probabilities of the type Eq.(5.7) and (5.8).

First consider Fig.21. Columns A to D refer to the parent CB net and columns F to I to the QB net. Column A for the CB net (F for the QB net) gives the identity of the occupation number $\underline{n}$ in Eqs.(5.5) and (5.6). Columns B and C for the CB net (G and H for the QB net) give the probabilities that $\underline{n} = 0$ and $\underline{n} = 1$, respectively, in light of the evidence. Column D for the CB net (I for the QB net) gives the quantity f_{qna} defined by Eq.(4.15). Note that for evidence-case 10, there was no output, because the computer program detected a contradiction. (In case 10, we were assuming that $n_{z+} = n_{z-} = 0$, which is impossible since $n_{z+} + n_{z-} = 1$.). More interesting is the evidence-case 4. In this case, for the QB net, column I indicates that $f_{qna} \neq 1$ for the distributions $P(n_{z+}|\underline{n}_{u+} = 0)$ and $P(n_{z-}|\underline{n}_{u+} = 0)$.[13]

Next consider Fig.22. Columns A to F refer to the parent CB net and columns H to M to the QB net. Column A for the CB net (H for the QB net) gives the identity of the occupation numbers $\underline{n}$ and $\underline{n}'$ in Eqs.(5.7) and (5.8). Columns B to E for the CB net (I to L for the QB net) give the probabilities that $(\underline{n}, \underline{n}') = (0,0), (0,1), (1,0), (1,1)$, respectively. Column F for the CB net (M for the QB net) gives f_{qna}. For example, in Fig.22 we see that for evidence-case 4, for the QB net, all the probability distributions except $P(n_{u+}, n_{u-}|\underline{n}_{u+} = 0)$ have $f_{qna} \neq 1$.[13]

(b) EXPERIMENTS WITH THREE STERN-GERLACH MAGNETS

We will consider 7 configurations with 3 Stern-Gerlach magnets: Figs.23 to 29.

For Fig.23, define

$$
\begin{aligned}
FI_1 &= A(\pi_1)\ , \\
&\quad A(\pi_1) = \psi_{10}\ , \\
FI_2 &= A(\pi_2)\ , \\
&\quad A(\pi_2) = \langle -_v|+_z\rangle\psi_{01}\ , \\
FI_3 &= A(\pi_3)\ , \\
&\quad A(\pi_3) = \langle -_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ , \\
FI_4 &= A(\pi_4)\ , \\
&\quad A(\pi_4) = \langle +_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ .
\end{aligned}
\tag{5.10}
$$

It is easy to show that $\sum_{j=1}^{4}|FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

For Fig.24, define

$$
\begin{aligned}
FI_1 &= A(\pi_1) + A(\pi_2)\ , \\
&\quad A(\pi_1) = \langle -_v|-_z\rangle\psi_{10}\ , \\
&\quad A(\pi_2) = \langle -_v|+_z\rangle\psi_{01}\ , \\
FI_2 &= A(\pi_3) + A(\pi_4)\ , \\
&\quad A(\pi_3) = \langle -_u|+_v\rangle\langle +_v|-_z\rangle\psi_{10}\ , \\
&\quad A(\pi_4) = \langle -_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ , \\
FI_3 &= A(\pi_5) + A(\pi_6)\ , \\
&\quad A(\pi_5) = \langle +_u|+_v\rangle\langle +_v|-_z\rangle\psi_{10}\ , \\
&\quad A(\pi_6) = \langle +_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ .
\end{aligned}
\tag{5.11}
$$

It is easy to show that $\sum_{j=1}^{3}|FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

For Fig.25, define

$$
\begin{aligned}
FI_1 &= A(\pi_1)\ , \\
&\quad A(\pi_1) = \psi_{10}\ , \\
FI_2 &= A(\pi_2) + A(\pi_3)\ , \\
&\quad A(\pi_2) = \langle -_u|-_v\rangle\langle -_v|+_z\rangle\psi_{01}\ , \\
&\quad A(\pi_3) = \langle -_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ , \\
FI_3 &= A(\pi_4) + A(\pi_5)\ , \\
&\quad A(\pi_4) = \langle +_u|-_v\rangle\langle -_v|+_z\rangle\psi_{01}\ , \\
&\quad A(\pi_5) = \langle +_u|+_v\rangle\langle +_v|+_z\rangle\psi_{01}\ .
\end{aligned}
\tag{5.12}
$$

It is easy to show that $\sum_{j=1}^{3}|FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

For Fig.26, define

$$
\begin{aligned}
FI_1 &= A(\pi_1) + A(\pi_2) + A(\pi_3) + A(\pi_4) \ , \\
A(\pi_1) &= \langle -_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\
A(\pi_2) &= \langle -_u | -_v \rangle \langle -_v | -_z \rangle \psi_{10} \ , \\
A(\pi_3) &= \langle -_u | +_v \rangle \langle +_v | +_z \rangle \psi_{01} \ , \\
A(\pi_4) &= \langle -_u | -_v \rangle \langle -_v | +_z \rangle \psi_{01} \ , \\
FI_2 &= A(\pi_5) + A(\pi_6) + A(\pi_7) + A(\pi_8) \ , \\
A(\pi_5) &= \langle +_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\
A(\pi_6) &= \langle +_u | -_v \rangle \langle -_v | -_z \rangle \psi_{10} \ , \\
A(\pi_7) &= \langle +_u | +_v \rangle \langle +_v | +_z \rangle \psi_{01} \ , \\
A(\pi_8) &= \langle +_u | -_v \rangle \langle -_v | +_z \rangle \psi_{01} \ .
\end{aligned}
\tag{5.13}
$$

It is easy to show that $\sum_{j=1}^{2} |FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

For Fig.27, define

$$
\begin{aligned}
FI_1 &= A(\pi_1) \ , \\
A(\pi_1) &= \langle -_v | -_z \rangle \psi_{10} \ , \\
FI_2 &= A(\pi_2) \ , \\
A(\pi_2) &= \langle +_v | -_z \rangle \psi_{10} \ , \\
FI_3 &= A(\pi_3) \ , \\
A(\pi_3) &= \langle -_u | +_z \rangle \psi_{01} \ , \\
FI_4 &= A(\pi_4) \ , \\
A(\pi_4) &= \langle +_u | +_z \rangle \psi_{01} \ .
\end{aligned}
\tag{5.14}
$$

It is easy to show that $\sum_{j=1}^{4} |FI_j|^2 = 1$, so Eq.(4.3) is satisfied by this net.

For Fig.28, define

$$
\begin{aligned}
FI_1 &= A(\pi_1) \ , \\
A(\pi_1) &= \langle -_v | -_z \rangle \psi_{10} \ , \\
FI_2 &= A(\pi_2) + A(\pi_3) \ , \\
A(\pi_2) &= e^{i\xi} \langle +_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\
A(\pi_3) &= \langle +_u | +_z \rangle \psi_{01} \ , \\
FI_3 &= A(\pi_4) + A(\pi_5) \ , \\
A(\pi_4) &= e^{i\xi} \langle -_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\
A(\pi_5) &= \langle -_u | +_z \rangle \psi_{01} \ .
\end{aligned}
\tag{5.15}
$$

Note that we have included a phase factor $e^{i\xi}$ which can be thought of as arising from the Stern-Gerlach node u when its input n_{v+} equals 1 (or from node v when its output n_{v+} equals 1, or from node z when its output n_{z-} equals 1). Alternatively, the phase factor may be though of as arising from a phase shifter node, not pictured in Fig.28, located in the middle of the arrow labelled n_{v+}. If $\xi = 0$, then $\sum_{j=1}^{3} |FI_j|^2 \neq 1$ and therefore Eq.(4.3) is violated. On the other hand, if

$$
e^{i\xi} = i \frac{\psi_{01} \psi_{10}^*}{|\psi_{01} \psi_{10}^*|} \ ,
\tag{5.16}
$$

then $\sum_{j=1}^{3} |FI_j|^2 = 1$.[14]

For Fig.29, define

$$\begin{aligned} FI_1 &= A(\pi_1) + A(\pi_2) + A(\pi_3) \ , \\ A(\pi_1) &= \langle +_u | +_z \rangle \psi_{01} \ , \\ A(\pi_2) &= e^{i\xi} \langle +_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\ A(\pi_3) &= e^{i\xi} \langle +_u | -_v \rangle \langle -_v | -_z \rangle \psi_{10} \ , \\ FI_2 &= A(\pi_4) + A(\pi_5) + A(\pi_6) \ , \\ A(\pi_4) &= \langle -_u | +_z \rangle \psi_{01} \ , \\ A(\pi_5) &= e^{i\xi} \langle -_u | +_v \rangle \langle +_v | -_z \rangle \psi_{10} \ , \\ A(\pi_6) &= e^{i\xi} \langle -_u | -_v \rangle \langle -_v | -_z \rangle \psi_{10} \ . \end{aligned} \tag{5.17}$$

It is easy to show that $\sum_{j=1}^{2} |FI_j|^2 = 1$ for any value of ξ. However, we believe the correct choice of ξ to be the one given by Eq.(5.16). For this choice, $P(\underline{n}_{u-} = 1 | \underline{n}_{v-} = 0)$ is the same for Figs.28 and 29. And this is what we expect. Otherwise, empty De Broglie waves could influence the outcome of an experiment (and thus could be detected), which does not appear to be the case experimentally.

For the QB nets of Figs.23 to 29, and for their parent CB nets, for $\underline{n}$, $\underline{n}'$, $\underline{m}$ and $\underline{m}' \in \{\underline{n}_{u\pm}, \underline{n}_{v\pm}, \underline{n}_{z\pm}\}$, for all n, n', m and $m' \in \{0, 1\}$, we calculated the conditional probabilities of Eqs.(5.5) to (5.8).

The table of Fig.30 is an evidence-case file that gives the sets of evidence that were considered for the nets of Figs.23 to 29. For example, in case 2 (the row that starts with a 2), we assumed $n_{z+} = 0$, whereas the values of the remaining occupation numbers $(n_{u\pm}, n_{v\pm}, n_{z-})$ were assumed to be unknown.

To get numerical values for the probabilities associated with the nets of Figs.23 to 29, particular values for $\psi_{01}, \psi_{10}, \theta_u, \theta_v, \theta_z$ were assumed.

For the tree graphs Figs.23 and 27, and also for the non-tree graph Fig.28, we found that the QB net and its parent CB net yielded identical probability distributions.

APPENDIX A. CONDITIONAL PROBABILITIES FOR FUZZY MEASUREMENTS

In this appendix, we will generalize the classical and quantum mechanical definitions of conditional probability Eqs.(4.7) and (4.14) so as to include either sharp or fuzzy hypotheses and pieces of evidence.

The following simple results from set theory are relevant. Given any finite set S, we will denote the number of elements in S by $|S|$. Given two sets R and S, we define the *direct product set* $R \times S$ by

$$R \times S = \{(x, y) | x \in R, y \in S\} \ . \tag{A.1}$$

$R \times S$ is also denoted by the *vector of sets* (R, S). Given sets R_1, R_2, S_1, S_2, it is easy to show that (see Fig.A.1)

$$(R_1 \times R_2) \cap (S_1 \times S_2) = (R_1 \cap S_1) \times (R_2 \cap S_2) \; . \tag{A.2}$$

Therefore, $(R_1 \times R_2) \cap (S_1 \times S_2) = \phi$ if and only if $R_1 \cap S_1 = \phi$ or $R_2 \cap S_2 = \phi$. Given any set S and any integer n, we will denote by S^n the product $S \times S \times \cdots \times S$ of n copies of S.

Suppose $Q_\alpha \subset Z_{0+}$ for each α. If $\Gamma' = \{\alpha_1, \alpha_2, \cdots, \alpha_{|\Gamma'|}\} \subset \Gamma$, we define the vector of sets or direct product set $(Q_\cdot)_{\Gamma'}$ by $(Q_\cdot)_{\Gamma'} = (Q_{\alpha_1}, Q_{\alpha_2}, \cdots, Q_{\alpha_{|\Gamma'|}})$. Equivalently, one may write $(Q_\cdot)_{\Gamma'} = Q_{\alpha_1} \times Q_{\alpha_2} \times \cdots \times Q_{\alpha_{|\Gamma'|}}$. Note that $(Q_\cdot)_{\Gamma'} \subset Z_{0+}^{|\Gamma'|}$. Sometimes, we will abbreviate $(Q_\cdot)_\Gamma$ by just $Q_\cdot$. Two direct product sets $(R_\cdot)_{|\Gamma'|}$ and $(S_\cdot)_{|\Gamma'|}$ are disjoint if and only if there exists an $\alpha \in \Gamma'$ such that $R_\alpha \cap S_\alpha = \phi$. We will need to consider collections $\mathcal{H} = \{H_\cdot^i | i = 1, 2 \cdots, |\mathcal{H}|\}$ of direct product sets $H_\cdot^i \subset Z_{0+}^{|\Gamma|}$. Such a collection will be said to be a *partition* of $Z_{0+}^{|\Gamma|}$ if any pair of distinct sets $H_\cdot^i$ and $H_\cdot^j$ is disjoint and $\cup_{i=1}^{|\mathcal{H}|} H_\cdot^i = Z_{0+}^{|\Gamma|}$.

We are interested in $P(\underline{n}_\cdot \in H_\cdot | \underline{n}_\cdot \in E_\cdot)$, i.e., the probability that $\underline{n}_\alpha \in H_\alpha$ for all $\alpha \in \Gamma$ given that $\underline{n}_\beta \in E_\beta$ for all $\beta \in \Gamma$. $H_\alpha \subset Z_{0+}$ is the *hypothesis for* $\underline{n}_\alpha$, and $E_\beta \subset Z_{0+}$ is the *evidence for* $\underline{n}_\beta$. If H_α contains only one element of Z_{0+}, then we say that the hypothesis for $\underline{n}_\alpha$ is *sharp*, whereas if H_α contains more that one element of Z_{0+}, we say that the hypothesis for $\underline{n}_\alpha$ is *fuzzy*. Analogously, the number of elements in E_β determines whether the evidence for $\underline{n}_\beta$ is sharp or fuzzy.

For any set S, define the *indicator function* $1_S(x)$ by

$$1_S(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases} \; . \tag{A.3}$$

Consider first a classical probability distribution $P(n_\cdot)$. One defines

$$P(\underline{n}_\cdot \in H_\cdot | \underline{n}_\cdot \in E_\cdot) = \frac{\sum_{n_\cdot \in H_\cdot \cap E_\cdot} P(n_\cdot)}{\sum_{n_\cdot \in E_\cdot} P(n_\cdot)} = \frac{\sum_{n_\cdot} P(n_\cdot) \prod_{\alpha \in \Gamma} 1_{H_\alpha \cap E_\alpha}(n_\alpha)}{\sum_{n_\cdot} P(n_\cdot) \prod_{\alpha \in \Gamma} 1_{E_\alpha}(n_\alpha)} \; . \tag{A.4}$$

To write the last equation more succinctly, it is convenient to define the *filter function* $f_{Q_\cdot}(n_\cdot)$, for any direct product set $Q_\cdot \subset Z_{0+}^{|\Gamma|}$, by

$$f_{Q_\cdot}(n_\cdot) = \prod_{\alpha \in \Gamma} 1_{Q_\alpha}(n_\alpha) \; . \tag{A.5}$$

It is also convenient to define the *characteristic functional* $\chi_c[K]$ of any function $K(n_\cdot)$ of $n_\cdot$ by

$$\chi_c[K] = \sum_{n_\cdot} P(n_\cdot) K(n_\cdot) \; . \tag{A.6}$$

(The c subscript in χ_c stands for "classical"). Now Eq.(A.4) can be written succinctly as

$$P(\underline{n}_\cdot \in H_\cdot | \underline{n}_\cdot \in E_\cdot) = \frac{\chi_c[f_{H_\cdot \cap E_\cdot}]}{\chi_c[f_{E_\cdot}]} \ . \tag{A.7}$$

Note that $f_{H_\cdot \cap E_\cdot} = f_{H_\cdot} f_{E_\cdot}$. Note that if $\mathcal{H} = \{H_\cdot^i | i = 1, 2, \cdots, |\mathcal{H}|\}$ is a partition of $Z_{0+}^{|\Gamma|}$, then Eq.(A.7) satisfies

$$\sum_{i=1}^{|\mathcal{H}|} P(\underline{n}_\cdot \in H_\cdot^i | \underline{n}_\cdot \in E_\cdot) = 1 \ . \tag{A.8}$$

Now consider a quantum mechanical probability amplitude $A(n_\cdot)$. We define the *characteristic functional* $\chi[K]$ by

$$\chi[K] = \sum_{(n_\cdot)_{\Gamma ext}} \left| \sum_{(n_\cdot)_{\Gamma int}} A(n_\cdot) K(n_\cdot) \right|^2 \ . \tag{A.9}$$

Suppose $\mathcal{H} = \{H_\cdot^i | i = 1, 2, \cdots, |\mathcal{H}|\}$ is a partition of $Z_{0+}^{|\Gamma|}$. In quantum mechanics, the conditional probability that $\underline{n}_\cdot \in H_\cdot^i$ given that $\underline{n}_\cdot \in E_\cdot$, depends on the choice of partition $\mathcal{H}$. We define this probability by

$$P_{\mathcal{H}}(\underline{n}_\cdot \in H_\cdot^i | \underline{n}_\cdot \in E_\cdot) = \frac{\chi[f_{H_\cdot^i \cap E_\cdot}]}{\sum_{j=1}^{|\mathcal{H}|} \chi[f_{H_\cdot^j \cap E_\cdot}]} \ . \tag{A.10}$$

Equation (A.8) is trivially satisfied by Eq.(A.10).

APPENDIX B. CONDITIONAL PROBABILITIES EXPRESSED IN TERMS OF PATH SUMS

In this appendix, we will express the classical and quantum mechanical definitions Eqs.(A.7) and (A.10) of conditional probability in terms of sums over paths. Simple examples of the results of this appendix may be found in Section 5 of this paper.

Let $P(n_\cdot)$ and $A(n_\cdot)$ be the values of a CB net and a QB net, respectively. Suppose that $\mathcal{P}(n_\cdot) = P(n_\cdot)$ if a CB net is being considered and $\mathcal{P}(n_\cdot) = A(n_\cdot)$ if a QB net is.

Let Π be the set of all possible *paths*. Π is defined so that there is exactly one $\pi \in \Pi$ for each $n_\cdot$ that satisfies $\mathcal{P}(n_\cdot) \neq 0$. Thus, there is a one to one onto map $n_\cdot(\pi)$ from Π into $\{n_\cdot \ | \mathcal{P}(n_\cdot) \neq 0\}$.

For any $\Gamma' \subset \Gamma$, if $m_\cdot$ is such that $(m_\cdot)_{\Gamma'} = (n_\cdot)_{\Gamma'}$, call $m_\cdot$ an *extension* of $(n_\cdot)_{\Gamma'}$. Let Σ be the set of all possible *final states*. Σ is defined so that there is

exactly one $\sigma \in \Sigma$ for each $(n.)_{\Gamma^{ext}}$ for which there exists an extension $n.$ such that $\mathcal{P}(n.) \neq 0$. Thus, there is a one to one onto map from Σ into $\{(n.)_{\Gamma^{ext}} | \mathcal{P}(n.) \neq 0\}$.

Define the *partition function* Z to be a function from Π to Σ such that $Z(\pi) = \sigma$ if the path π has σ as final state. For each $\sigma \in \Sigma$, let $C(\sigma)$ be the set of all paths π that have σ as final state; i.e., $C(\sigma) = \{\pi \in \Pi | Z(\pi) = \sigma\}$. Clearly, the sets (equivalence classes) $C(\sigma)$ and $C(\sigma')$ are disjoint when $\sigma \neq \sigma'$, and $\cup_{\sigma \in \Sigma} C(\sigma) = \Pi$.

For each $\pi \in \Pi$, define $\overline{\mathcal{P}}(\pi)$ by $\overline{\mathcal{P}}(\pi) = \mathcal{P}(n.(\pi))$.

For any direct product set $Q. \subset Z_{0+}^{|\Gamma|}$, define the filter function $\overline{f}_{Q.}(\pi)$ by

$$\overline{f}_{Q.}(\pi) = \prod_{\alpha \in \Gamma} 1_{Q_\alpha}(n_\alpha(\pi)) \; . \tag{B.1}$$

Classically, one defines the characteristic functional $\overline{\chi}_c[K]$ of any function $K(\pi)$ of π by

$$\overline{\chi}_c[K] = \sum_{\sigma \in \Sigma} \sum_{\pi \in C(\sigma)} \overline{P}(\pi) K(\pi) \; . \tag{B.2}$$

Then

$$P(\underline{n}. \in H. | \underline{n}. \in E.) = \frac{\overline{\chi}_c(\overline{f}_{H. \cap E.})}{\overline{\chi}_c(\overline{f}_{E.})} \; . \tag{B.3}$$

Quantum mechanically, one defines the characteristic functional $\overline{\chi}[K]$ by

$$\overline{\chi}[K] = \sum_{\sigma \in \Sigma} \left| \sum_{\pi \in C(\sigma)} \overline{A}(\pi) K(\pi) \right|^2 \; . \tag{B.4}$$

If $\mathcal{H} = \{H_{\cdot}^i | i = 1, 2, \cdots, |\mathcal{H}|\}$ is a partition of $Z_{0+}^{|\Gamma|}$, then one defines

$$P_{\mathcal{H}}(\underline{n}. \in H_{\cdot}^i | \underline{n}. \in E.) = \frac{\overline{\chi}(\overline{f}_{H_{\cdot}^i \cap E.})}{\sum_{j=1}^{|\mathcal{H}|} \overline{\chi}(\overline{f}_{H_{\cdot}^j \cap E.})} \; . \tag{B.5}$$

APPENDIX C. QB NET FOR SINGLE PARTICLE FEYNMAN PATH INTEGRAL

In this appendix, we will present a QB net which yields the Feynman path integral[15] that in turn yields the Schroedinger equation for a single mass particle in an arbitrary potential.

We begin by restricting positions x to lie in the interval ("box") $[\frac{-L}{2}, \frac{L}{2}]$, where L is much larger that any other length (including the particle's position coordinate) occurring in the problem. Then we divide the box into subintervals of

infinitesimal length Δx, and call the midpoints of these subintervals x_s with $s \in \{0, \pm 1, \pm 2, \cdots, \pm N_x/2\} = Z_x$. Similarly, we restrict times t to lie in the interval $[0, T]$. Then we divide the interval $[0, T]$ into subintervals of infinitesimal length Δt, and call the midpoints of these subintervals t_i with $i \in \{0, 1, 2, \cdots, N_t\} = Z_t$.

Henceforth, for any function $f(x_s)$, we will use $f(\cdot)$ to represent the vector whose sth component is $f(x_s)$. For example, $\delta_{x_r}(\cdot)$ will represent the vector whose sth component $\delta_{x_r}(x_s)$ is 1 if $r = s$ and zero otherwise.

As in Fig.C.1, we assign a node and a random variable $\underline{n}(x_s, t_i)$ to all space-time lattice points (x_s, t_i) with $s \in Z_x$ and $i \in Z_t - \{0\}$ and to the lattice point (x_0, t_0). (Thus, at time $t_0 = 0$, only position x_0 gets a node). We draw arrows pointing from each node to all nodes occurring a time Δt later. We also draw external arrows pointing out of each node at time t_{N_t}. The net starts at time t_0 with a single root node at position x_0, and it ends at time t_{N_t} with external nodes at each position x_s for all $s \in Z_x$. The random variables $\underline{n}(x_s, t_i)$ are occupation numbers that assume values $n(x_s, t_i) \in Z_{0+}$. These occupation numbers specify states

$$|n(x_s, t_i)\rangle = \frac{[a^\dagger(x_s, t_i)]^{n(x_s, t_i)}}{\sqrt{n(x_s, t_i)!}} |0\rangle \ . \tag{C.1}$$

From the states of Eq. (C.1), one can form states $|n(\cdot, t_i)\rangle$ defined by

$$|n(\cdot, t_i)\rangle = \Pi_{s \in Z_x} |n(x_s, t_i)\rangle \ . \tag{C.2}$$

Figure C.2 shows the input and output arrows for a single node (x_s, t_i) of our net. This node is assigned a value

$$A[n(x_s, t_i) | n(\cdot, t_{i-1})] = \langle n(x_s, t_i) | \Omega | n(\cdot, t_{i-1}) \rangle \ , \tag{C.3}$$

where Ω is a second quantized operator.

In general, the theory characterized by this QB net is a multi-particle quantum field theory. In fact, $n(x_s, t_i)$ is the classical field $\phi(x_s, t_i)$ that is commonly used to define Feynman integrals in quantum field theories. In this appendix, we will consider a single particle. Thus, for all i, $\sum_{s \in Z_x} n(x_s, t_i) = 1$ and $n(\cdot, t_i) = \delta_{x_r}(\cdot)$ for some $r \in Z_x$. For a single particle, the root node (x_0, t_0) will be taken to have amplitude 1 for $n(x_0, t_0) = 1$ and amplitude 0 for $n(x_0, t_0) = 0$. All other nodes will be taken to have the value $A[n(x_s, t_i) | n(\cdot, t_{i-1})]$ that is specified by the table of Fig.C.3. The quantity $\alpha_{s,r}$ in Fig.C.3 is given by

$$\alpha_{s,r} = \langle x_s | e^{\frac{-i}{\hbar} \Delta t \ H} | x_r \rangle \ . \tag{C.4}$$

In this last equation, $\hbar$ is Planck's constant; H is the single particle, first quantized Hamiltonian; $|x_r\rangle$ for $r \in Z_x$ are the position eigenvectors, normalized so that $\langle x_s | x_r \rangle = \delta_{s,r}$ and $\sum_{r \in Z_x} |x_r\rangle\langle x_r| = 1$, where $\delta_{s,r}$ is the Kronecker delta. Equation (4.1) is trivially satisfied by the table of Fig.C.3. As for Eq.(4.3), it follows from the identity

$$\sum_{s\in Z_x} |\langle x_s| e^{\frac{-i}{\hbar}(t_{N_t}-t_0)H}|x_0\rangle|^2 = 1 \ . \tag{C.5}$$

For $H = \frac{p^2}{2m} + V(x,t)$, one may write

$$\alpha_{s,r} = \exp\left\{\frac{-i\Delta t}{\hbar}\left[(\frac{-\hbar^2}{2m})(\frac{d}{dx_s})^2 + V(x_s,t)\right]\right\}\Delta x\delta(x_s - x_r) \ , \tag{C.6}$$

where $\delta(\cdot)$ is the Dirac delta function. If one were to expand the exponential in the last equation, terms in which $(\frac{d}{dx_s})^2$ acted only on the delta function would be much larger than those in which it acted on $V(x_s,t)$. Thus, one may approximate $\alpha_{s,r}$ by keeping only those terms in which $(\frac{d}{dx_s})^2$ does not act on $V(x_s,t)$. One then gets

$$\alpha_{s,r} \approx (\alpha_{s,r})_{free}\, e^{\frac{-i}{\hbar}\Delta t V(x_s,t)} \ , \tag{C.7}$$

where

$$(\alpha_{s,r})_{free} = \Delta x \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(x_s-x_r) - \frac{i\Delta t}{\hbar}(\frac{\hbar^2 k^2}{2m})} \ . \tag{C.8}$$

The Gaussian integral in Eq.(C.8) is easily performed. One obtains

$$\alpha_{s,r} = \sqrt{\frac{-i\Delta\theta}{\pi}} e^{\frac{i}{\hbar}\Delta t \mathcal{L}_{s,r}} \ , \tag{C.9}$$

where

$$\Delta\theta = (\frac{\Delta t}{\hbar})\frac{m}{2}(\frac{\Delta x}{\Delta t})^2 \ , \tag{C.10}$$

and[16]

$$\mathcal{L}_{s,r} = \frac{m}{2}\left(\frac{x_s - x_r}{\Delta t}\right)^2 - V(x_s,t) \ . \tag{C.11}$$

In general, $|\alpha_{s,r}| << 1$. This is why. The phase $\frac{\Delta t \mathcal{L}_{s,r}}{\hbar}$ of $\alpha_{s,r}$ must not show any granularity when x_s and x_r range over their possible discrete values. Otherwise, the outcome of experiments that depended on this phase would depend on Δx. When x_s goes from x_r to $x_r + \Delta x$, the phase $\frac{\Delta t \mathcal{L}_{s,r}}{\hbar}$ jumps by $\Delta\theta$. If this jump is small, then so is $|\alpha_{s,r}| = \sqrt{\frac{\Delta\theta}{\pi}}$.

Consider Fig.C.4, which presents a typical evaluation of the net of Fig.C.1. We have blackened those nodes whose state $\underline{n}(x_s,t_i)$ is 1. Arrows coming out of the blackened nodes are also imagined to lie in state 1. We show only those arrows lying in state 1; all other arrows, i.e., those lying in state 0, are not shown. The value of each node (the infinitesimal complex amplitude $\alpha_{s,r}$ for the blackened nodes and approximately 1 for the unblackened ones) has been placed next to it. If one connects the blackened nodes one obtains a path. The value of the whole net equals

the product of the node values along this path. Thus, the sum over all possible evaluations of Fig.C.1 with the same final node (at time T) blackened may be viewed as a sum over all paths with the same final state. This latter path sum is precisely the familiar Feynman path integral that yields the Shroedinger equation for a single particle in an arbitrary potential.[15]

Before concluding this appendix, let us consider the classical limit of the QB net of Fig.C.1.

Recall the method of obtaining the particle's classical equation of motion using the action. The action $S[x(\cdot)]$ for a path $x(t)$ is

$$S[x(\cdot)] = \int_0^T dt\ \mathcal{L}\ , \tag{C.12}$$

where the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{m}{2}[\dot{x}(t)]^2 - V[x(t), t]\ . \tag{C.13}$$

Let $\delta x(t)$ be any smooth function of time. For any functional $F[x(\cdot)]$ of a function $x(t)$, define δF by

$$\delta F = F[x(\cdot) + \delta x(\cdot)] - F[x(\cdot)]\ . \tag{C.14}$$

The equation of motion is obtained by setting $\delta S = 0$ to first order in $\delta x(t)$ for all $\delta x(\cdot)$ with $\delta x(0) = \delta x(T) = 0$. One gets

$$\delta S = \int_0^T dt\ \left(-m\ddot{x}(t) - \frac{\partial V}{\partial x}[x(t), t]\right) \delta x(t) = 0\ , \tag{C.15}$$

for arbitrary $\delta x(t)$, which implies

$$m\ddot{x} = -\frac{\partial V}{\partial x}\ . \tag{C.16}$$

The classical path $x_{cl}(t)$ is defined as the solution to Eq.(C.16) for given $x(0)$ and $x(T)$. The classical action is defined by $S_{cl} = S[x_{cl}(\cdot)]$. When $\delta S = 0$, $x(t) = x_{cl}(t)$ and $S[x(\cdot)] = S_{cl}$.

If we restrict $x(t)$ to take on values in the set $\{x_r | r \in Z_x\}$, then to the path $x(\cdot)$ there corresponds a sequence of integers $r(t_i)$ for each $i \in Z_t$, where $x(t_i) = x_{r(t_i)}$. The path $x(t)$ connects a string of nodes $(x_{r(t_i)}, t_i)$ (the blackened nodes in Fig.C.4). The value of the QB net Fig.C.1 is the the product of the values $\alpha_{r(t_i),r(t_{i-1})}$ of these nodes. From Eqs.(C.11), (C.12) and (C.13),

$$S[x(\cdot)] = \sum_{i \in Z_t} \Delta t \mathcal{L}_{r(t_i),r(t_{i-1})}\ . \tag{C.17}$$

Thus,

$$\prod_{i\in Z_t} \alpha_{r(t_i),r(t_{i-1})} = \left(\frac{-i\Delta\theta}{\pi}\right)^{\frac{N_t}{2}} e^{\frac{i}{\hbar}S[x(\cdot)]} \ . \tag{C.18}$$

Probabilities are calculated by summing the weights Eq.(C.18) for all possible paths $x(\cdot)$ with the same $x(T)$ and taking the magnitude squared of the sum. The paths which interfere constructively and therefore contribute the most to such probabilities are those for which $\left|\frac{\delta S}{\hbar}\right| << 1$, i.e., those for which the phase change $\frac{S}{\hbar}$ accumulated over the time interval $(0,T)$ is nearly stationary under deformations $\delta x(t)$ of the path. In the classical limit, which occurs when $\hbar \to 0$, $\left|\frac{\delta S}{\hbar}\right| << 1$ means $\delta S \to 0$.

Note that the classical limit of the QB net Fig.C.1 is not its parent CB net. With the parent CB net, paths do not interfere, whereas with the QB net in the classical limit, there is so much destructive interference between paths that all path except the classical one cancel each other out. With the parent CB net, equal weight is given to smooth paths like those a classical particle would follow and to jagged paths with unbounded variations. With the QB net in the classical limit, paths with unbounded variations cancel each other out.

APPENDIX D. SPIN $\frac{1}{2}$ STATES

Our conventions for spin $\frac{1}{2}$ states are as follows.

Let $\hat{u}$ be a unit 3-dimensional vector characterized by angles (θ,ϕ) (see Fig.D.1). Thus,

$$\hat{u} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \ . \tag{D.1}$$

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be the vector of Pauli matrices, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{D.2}$$

If $|+_u\rangle$ and $|-_u\rangle$ are defined by

$$\sigma\cdot\hat{u}|+_u\rangle = |+_u\rangle \ , \ \ \sigma\cdot\hat{u}|-_u\rangle = -|-_u\rangle \ , \tag{D.3}$$

then one can show that

$$|+_u\rangle = \begin{pmatrix} CE^* \\ SE \end{pmatrix} \ , \quad |-_u\rangle = \begin{pmatrix} -SE^* \\ CE \end{pmatrix} \ , \tag{D.4}$$

where

$$S = \sin\frac{\theta}{2} \ , \quad C = \cos\frac{\theta}{2} \ , \quad E = e^{\frac{i\phi}{2}} \ . \tag{D.5}$$

For example, if $\theta = \phi = 0$, one gets

$$|+_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad |-_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{D.6}$$

By Eqs.(D.4) and (D.5), if $\hat{u}$ and $\hat{u'}$ are unit vectors characterized by angles (θ_u, ϕ_u) and $(\theta_{u'}, \phi_{u'})$, respectively, and if $\phi_u = \phi_{u'} = 0$, then

$$\langle +_{u'}|+_u\rangle = \langle -_{u'}|-_u\rangle = \cos\left(\frac{\theta_{u'} - \theta_u}{2}\right) \ , \tag{D.7}$$

$$\langle +_{u'}|-_u\rangle = -\langle -_{u'}|+_u\rangle = \sin\left(\frac{\theta_{u'} - \theta_u}{2}\right) \ . \tag{D.8}$$

For any unit vector $\hat{u}$, the antisymmetric state for two particles 1 and 2 is

$$|\psi_u^{ant}\rangle = \frac{1}{\sqrt{2}}\left[|+_u\rangle_1|-_u\rangle_2 - |-_u\rangle_1|+_u\rangle_2\right] \ . \tag{D.9}$$

($|\psi_u^{ant}\rangle$ has zero total angular momentum and is thus also called the singlet state). By expressing the states $|\pm_u\rangle$ on the right hand side of Eq.(D.9) in terms of $|\pm_z\rangle$, it is easy to show that $|\psi_u^{ant}\rangle$ is invariant under rotations, i.e., it is independent of $\hat{u}$.

So far, our formulation has been a first quantized one. In a second quantized formulation, one defines annihilation operators $a_{u\sigma}$, for any unit vector $\hat{u}$ and for $\sigma \in \{+, -\}$. These annihilation operators must satisfy

$$[a_{u\sigma}, a_{u'\sigma'}]_+ = 0 \ , \tag{D.10}$$

$$[a_{u\sigma}, a^\dagger_{u'\sigma'}]_+ = \delta_{u,u'}\delta_{\sigma,\sigma'} \ , \tag{D.11}$$

where, for any two operators A and B, $[A, B]_+ = AB + BA$. States in the first quantized formulation are mapped in a 1-1 fashion into states in the second quantized formulation. For example,

$$|+_u\rangle \to a^\dagger_{u+}|0\rangle \ , \tag{D.12}$$

$$|-_u\rangle \to a^\dagger_{u-}|0\rangle \ , \tag{D.13}$$

$$|\psi^{ant}\rangle \to a^\dagger_{u+}a^\dagger_{u-}|0\rangle \ . \tag{D.14}$$

From Eqs.(D.4), (D.12) and (D.13), it follows that

$$\begin{pmatrix} a^\dagger_{u+} \\ a^\dagger_{u-} \end{pmatrix} = \begin{pmatrix} CE^* & SE \\ -SE^* & CE \end{pmatrix} \begin{pmatrix} a^\dagger_{z+} \\ a^\dagger_{z-} \end{pmatrix} . \tag{D.15}$$

(To check this last equation, just apply $|0\rangle$ from the right to both sides of the equation.)

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[11] More precisely, by $P(\underline{x} = x)$ we mean the probability associated with the set $\{\omega \in \Omega | \underline{x}(\omega) = x\}$, where Ω is the outcome space and the random variable $\underline{x}$ is a function from Ω to the set of all possible values x. Authors of mathematical texts (see, for example, *Modern Probability Theory and its Applications* (Wiley, 1960), by E. Parzen) often write $P(X = x)$, but using upper case for random variables and lower case for their possible values is impractical in Physics.

[12] For a review and original references dealing with the Clauser-Horne experiment and its Bell inequalities, see R.R. Tucci, J. of Mod. Physics B **.

[13] We describe as *pre-conditioning* a situation in which the evidence nodes ($\underline{m}$ and $\underline{m}'$ in Eqs.(5.5) to (5.8)) precede the hypothesis nodes ($\underline{n}$ and $\underline{n}'$ in Eqs.(5.5) to (5.8)). In general, failure to pre-condition does not imply $f_{qna} \neq 1$ (see, for example, tree nets). However, we observed that very often $f_{qna} \neq 1$ implies failure to pre-condition.

[14] The phase factor $e^{i\xi}$ does not appear to be a consequence of the azimuthal phase factors $e^{\frac{\phi}{2}}$ of Eq.(D.5): we've assumed the magnetic field directions $\hat{u}$, $\hat{v}$ and $\hat{z}$ to be coplanar, and therefore we may assume that $\phi_u = \phi_v = \phi_z = 0$. Suppose $\hat{a}, \hat{b}, \hat{c}$ are the particle travel directions inside the magnets with magnetic fields in the directions $\hat{u}, \hat{v}, \hat{z}$, respectively. ($\hat{a} \cdot \hat{u} = \hat{b} \cdot \hat{v} = \hat{c} \cdot \hat{u} = 0$). The phase factor $e^{i\xi}$ is still necessary even if $\hat{a}, \hat{b}, \hat{c}$ are assumed to be parallel, which is a stronger assumption than only assuming that $\hat{u}$, $\hat{v}$ and $\hat{z}$ are coplanar.

[15] R.P. Feynman, and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw Hill, 1965).

[16] It would have been just as good an approximation to use $V(x_r, t)$ instead of $V(x_s, t)$ in Eq.(C.11). For small Δt, if the kinetic energy term $\frac{1}{2m}(\frac{x_s - x_r}{\Delta t})^2$ is to contribute a finite phase to $\alpha_{s,r}$, x_s and x_r must remain close, and therefore $V(x_r, t) \approx V(x_s, t)$.

Figure CAPTIONS:

Fig.1 Node labelled by the random variable $\underline{x}_j$.

Fig.2 Possible CB nets with 2 nodes $\underline{x}$ and $\underline{y}$.

Fig.3 Possible CB nets with 3 nodes $\underline{x}$, $\underline{y}$ and $\underline{z}$.

Fig.4 Three-node cycle.

Fig.5 Proof that the joint probability represented by Fig.3d adds up to one (i.e., satisfies Eq.(2.4)). Summation over the states of an arrow is indicated by giving the arrow a double shaft.

Fig.6 Figure 6a shows the fully connected four node graph with its chronological labelling. By deforming Fig.6a into a topologically equivalent diagram, one obtains a more "stylized" version of the same thing, Fig.6b.

Fig.7 Chapman-Kolgomorov equation for a Markov chain $(\underline{x}, \underline{y}, \underline{z})$.

Fig.8 Generalization of Fig.7 to arbitrary random variables $\underline{x}, \underline{y}, \underline{z}$ that don't necessarily form a Markov chain.

Fig.9 CB net version of an AND gate.

Fig.10 $\underline{z}$ is a SUM node.

Fig.11 $\underline{z}$ is an IF-THEN node.

Fig.12 CB net for the Clauser-Horne experiment. This figure really represents 4 nets, one for each of the following possibilities: $(\theta_1, \theta_2) = (A, B), (A, B'), (A', B), (A', B')$.

Fig.13 CB net for an experiment like the usual Clauser-Horne experiment, except that here the measurement types θ_1 and θ_2 are selected at random.

Fig.14 CB nets associated with a random walk. Net Fig.14c is coarser than net Fig.14b, which is coarser than net Fig.14a.

Fig.15 Root node representing initial wavefunction.

Fig.16a Marginalizer node.

Fig.16b Phase Shifter node.

Fig.17 Nodes for Stern-Gerlach magnet with (a) one, and (b) two incoming modes.

Fig.18 QB net for an experiment with 2 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.19 QB net for an experiment with 2 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.20 Various sets of evidence that were considered for the nets Figs.18 and 19 with two Stern-Gerlach magnets.

Fig.21 One-hypothesis probabilities, for the net of Fig.19, for evidence-cases 1, 2, 4, 10 and 12. Columns A to D refer to the parent CB net and columns F to I to the QB net.

Fig.22 Two-hypothesis probabilities, for the net of Fig.19, for evidence-cases 1, 2, 4, 10 and 12. Columns A to F refer to the parent CB net and columns H to M to the QB net.

Fig.23 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.24 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.25 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.26 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.27 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.28 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.29 QB net for an experiment with 3 Stern-Gerlach magnets. (triangle=wavefunction, black-filled circle=marginalizer, white-filled circle=magnet).

Fig.30 Various sets of evidence that were considered for the nets Figs.23 to 29 with three Stern-Gerlach magnets.

Fig.A.1 Graphical proof of an identity about the intersection of two direct product sets.

Fig.C.1 "The fabric of spacetime". QB net that yields the Feynman path integral for a single mass particle in an arbitrary potential.

Fig.C.2 The input and output arrows for a single node (x_s, t_i) of the net Fig.C.1.

Fig.C.3 In the net Fig.C.1, the value $A[n(x_s, t_i)|n(\cdot, t_{i-1})]$ of the node (x_s, t_i).

Fig.C.4 A typical evaluation of the net of Fig.C.1. We have blackened those nodes whose state $\underline{n}(x_s, t_i)$ is 1. Arrows coming out of the blackened nodes are also imagined to lie in state 1. We show only those arrows lying in state 1; all other arrows, i.e., those lying in state 0, are not shown. The value of each node (the infinitesimal complex amplitude $\alpha_{s,r}$ for the blackened nodes and approximately 1 for the unblackened ones) has been placed next to it.

Fig.D.1 Definition of the angles θ and ϕ used in specifying the unit vector $\hat{u}$.