Chapter 4

DIELECTRIC PROPERTIES, ELECTRIC ENERGY AND FORCE

4.1   Introduction

So far, we have been analyzing various electrostatic problems by assuming an ideal (vacuum) medium characterized by the permittivity ϵ_0. Most capacitors are filled with dielectric materials having permittivities larger than ϵ_0. Coaxial cables are filled with Teflon as an insulating material between the inner and outer conductors. Teflon has a permittivity of approximately $\epsilon = 2\epsilon_0$. Even the permittivity of air is not exactly equal to ϵ_0, but slightly larger, $\epsilon \simeq 1.0004\epsilon_0$, because of polarization of air molecules.

In this Chapter, the origin of dielectric properties will be discussed. As we will see, dielectric properties are caused by electric dipoles either intrinsically present in molecules or induced by an external electric field. For example, the charge distribution in a water molecule is not completely symmetric because of the peculiar locations of two hydrogen atoms near the oxygen atom. (Fig. 4.1) A water molecule thus has a permanent dipole moment of its own. When water is placed in an electric field, dipoles are aligned along the field which reduces the field in water, as shown in
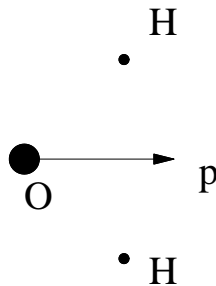

Figure 4-1: Permanent dipole moment of water molecule H_2O is caused by the shift of center of electron cloud relative to that of cneter of mass of the positive charges.
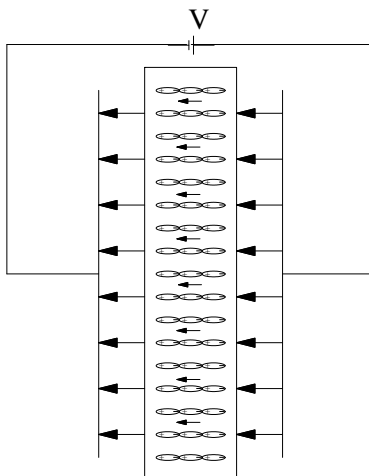
Figure 4-2: Dipoles placed in an external electric field are co-aligned with the field and the field in the dielectric becomes smaller than the external field by a factor $\varepsilon/\varepsilon_0$.

Fig. 4.2.

As an example of induced dielectric properties, let us consider a hydrogen atom. (Fig. 4.3) In the absence of an external electric field, electron distribution about the proton is symmetric, and the dipole moment is zero. However, when a hydrogen atom is placed in an electric field, electron distribution is shifted opposite to the electric field, and the charge distribution is no more symmetric. If the center of electron cloud is shifted by Δx from the proton, the dipole moment may be estimated from

$$p = e^2 \Delta x \tag{4.1}$$

Since Δx is proportional to the applied electric field, the dipole moment is also proportional to the field. This simple case thus indicates that materials having no intrinsic dipole moments can also behave as dielectric materials when placed in an electric field.

4.2   Potential due to Distributed Dipole Moments

In Chapter 2, we have seen that the potential due to a single dipole moment $\mathbf{p}$ located at the origin is given by

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \tag{4.2}$$

Furthermore, if many dipoles are continuously distributed with a dipole moment density $\mathbf{P}(\mathbf{r})$, the potential can be given in the following integral form,

$$\Phi_d(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' \tag{4.3}$$
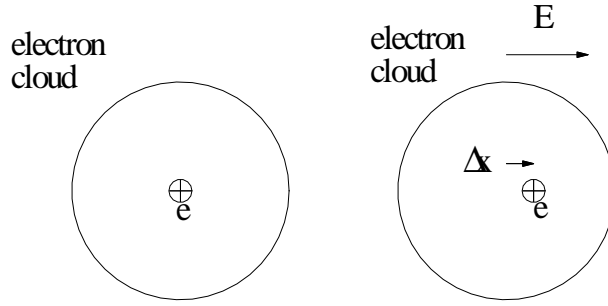
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Figure 4-3: Left: In a hydrogen atom, the center of electron cloud coincides with the proton and there is no dipole moment. Right: If placed in an external electric field, the center of elctron cloud is shifted from the proton and a dipole moment is induced in the direction of the external field.

Since the dimensions of the single dipole $\mathbf{p}$ are C·m, the dimensions of the dipole moment density $\mathbf{P}$ are C/m^2, which is formally identical to the dimensions of the surface charge density σ (C/m^2).

The integral in Eq. (4.3) can be rewritten as

$$\int \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$
$$= \int \left[\nabla_{r'} \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla_{r'} \cdot \mathbf{P}(\mathbf{r}') \right] dV' \tag{4.4}$$

since

$$\nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{4.5}$$

However, the first integral in Eq. (**??**) can be converted into a surface integral,

$$\int \nabla_{r'} \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' = \oint \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{S}' \tag{4.6}$$

which vanishes because the closed surface S' is at infinity where $\mathbf{P}(\mathbf{r}')$ is certainly zero. Therefore, the potential due to a dipole moment distribution becomes

$$\Phi_d(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int \frac{\nabla_{r'} \cdot \mathbf{P}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{4.7}$$

Comparing with the familiar expression for the potential due to a distributed charge,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{4.8}$$

we conclude that the divergence of the dipole moment density

$$\rho_{eff} = -\nabla \cdot \mathbf{P} \quad (C/m^3) \tag{4.9}$$

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may be regarded as an effective charge density.

At this stage, it is convenient to classify electric charges depending on their freedom. Charges supplied from a power supply are called free charges, while charges bound to molecules are called bound charges. Bound charges are not free to move around as free charges. As long as microscopic electrodynamics is concerned, the electric field is uniquely determined from

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4.10}$$

in which ρ is the total charge, including both free and bound charges. In the presence of many dipoles, it is more convenient to separate the charge ρ into free and bound charges,

$$
\begin{aligned}
\rho_{\text{Total}} &= \rho_{\text{free}} + \rho_{\text{bound}} \\
&= \rho_{\text{free}} - \nabla \cdot \mathbf{P}
\end{aligned} \tag{4.11}
$$

Substituting this into Eq. (4.10), we obtain,

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{free}}}{\epsilon_0} - \frac{1}{\epsilon_0} \nabla \cdot \mathbf{P} \tag{4.12}$$

or

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}} \tag{4.13}$$

In practical applications, this form is more convenient since the free charge is the quantity that can be controlled by external means (such as power supplies).

Now we introduce a new vector defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.14}$$

Maxwell coined the name "displacement vector" for $\mathbf{D}$. Although Maxwell's physical picture for the vector $\mathbf{D}$ has been largely discarded, its name is still in use for traditional reasons. In some literature, $\mathbf{D}$ is called the electric flux density vector, in duality to the magnetic flux density vector, $\mathbf{B}$. Note that the vector $\mathbf{D}$ has dimensions of C/m^2 , that is, the same dimensions as the surface charge density.

As we will see in the following Section, the dipole moment density $\mathbf{P}$ in ordinary materials is proportional to the electric field $\mathbf{E}$. Therefore, in most cases, the displacement vector $\mathbf{D}$ becomes proportional to the electric field, and we may write

$$\mathbf{D} = \epsilon \mathbf{E} \tag{4.15}$$

The proportional constant ϵ is the permittivity. In contrast to the vacuum permittivity ϵ_0, which is a genuine constant, the permittivity in a material may vary from place to place, since the dipole moment density $\mathbf{P}$ is in general nonuniform. Furthermore, in an anisotropic dielectric material, the permittivity is in general a tensor, $\mathbf{D} = \boldsymbol{\varepsilon} \cdot \mathbf{E}$.

4.3 Qualitative Estimates of Dipole Moment Density P

In this Section, we will calculate the dipole moment density $\mathbf{P}$ of a material placed in an electric field. In a material consisting of molecules having no intrinsic dipole moments, dielectric properties are entirely due to the displacement of electrons relative to ions, as briefly explained in the Introduction. Most gases belong to this so-called "electronic polarization" category. For simplicity, let us consider a hydrogen gas. In the absence of external electric fields, the electron undergoes harmonic motion about the proton with a frequency ω_0. An order of magnitude estimate for this frequency can be made from the ionization energy, which is the energy required to split the electron off the proton. The ionization energy of a hydrogen atom is about 13.6 eV, or $13.6 \times 1.6 \times 10^{-19}$ J. Then, from

$$\hbar\omega_0 \simeq 13.6 \times 1.6 \times 10^{-19}$$

where $\hbar = h/2\pi = 1.06 \times 10^{-34}$ J·sec is the Planck's constant divided by 2π, we find $\omega_0 \simeq 2 \times 10^{16}$ rad/sec.

Harmonic motion of the electron in the absence of an electric field can be described by

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{4.16}$$

When an electric field is applied, the center of electron harmonic motion is displaced from the proton. We denote this displacement by Δx as indicated in Fig. 4.3. Since the force exerted on the electron by the electric field is $-e\mathbf{E}$, the acceleration is given by

$$-\frac{e}{m}\mathbf{E}$$

This dc acceleration must be counterbalanced by the shift Δx,

$$\omega_0^2 \Delta x = \frac{e}{m}E \tag{4.17}$$

Then,

$$\Delta x = \frac{e}{m\omega_0^2}E \tag{4.18}$$

Since the electron and proton charges are equal and opposite, the displacement Δx creates a dipole moment per hydrogen atom,

$$\mathbf{p} = e\Delta x = \frac{e^2}{m\omega_0^2}\mathbf{E} \tag{4.19}$$

If the hydrogen gas has n atoms per unit volume, the dipole moment density becomes

$$\mathbf{P} = \frac{Ne^2}{m\omega^2}\mathbf{E} \tag{4.20}$$

Substituting this into Eq. (4.14), we find,

$$
\begin{aligned}
\mathbf{D} &= \epsilon_0 \mathbf{E} + \frac{Ne^2}{m\omega_0^2}\mathbf{E} \\
&= \epsilon_0 \left(1 + \frac{Ne^2}{m\epsilon_0\omega_0^2} \right) \mathbf{E}
\end{aligned}
\tag{4.21}
$$

which defines the permittivity

$$
\epsilon = \epsilon_0 \left(1 + \frac{Ne^2}{m\epsilon_0\omega_0^2} \right)
\tag{4.22}
$$

Let us evaluate this quantity for a hydrogen gas at $0°$ C, 1 atmospheric pressure. The number of hydrogen atom per unit volume is about $n \simeq 5.4 \times 10^{25}/\text{m}^3$. Substituting $e = 1.6 \times 10^{-19}$ C, $m = 9.1 \times 10^{-31}$ kg, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m, and $\omega_0 = 2 \times 10^{16}/\text{s}$, we find

$$
\frac{ne^2}{m\epsilon_0\omega_0^2} \simeq 4 \times 10^{-4}
$$

and the correction to the permittivity is approximately

$$
\epsilon \simeq 1.0004\epsilon_0.
$$

This rough estimate compares favourably with the experimental value, $\epsilon \simeq 1.0003\epsilon_0$. Although the correction is small, the fact that the permittivities of gases (including air) are slightly larger than ϵ_0 causes observable effects, such as mirage and star twinkling. The blueness of the sky, and redness at sunset and dawn are also caused by air molecules. Note that the correction to the permittivity is proportional to the molecule density.

In contrast to gases, most liquids and solids consist of molecules having intrinsic permanent dipole moments. For example, a water molecule has a dipole moment (per molecule)

$$
p \simeq 6 \times 10^{-30} \text{ C} \cdot \text{m}
$$

Since the typical size of a molecule is of order 10^{-10} m, and the characteristic charge is $e = 1.6 \times 10^{-19}$ C, the value of the dipole moment is more or less what we expect.

In the absence of external electric fields, molecules are oriented randomly and no macroscopic electric effects occur. Randomness is due to thermal agitation. When an electric field is applied, dipoles tend to align themselves in the direction of electric field. Alignment will not be complete, however, because of thermal agitation. Since thermal randomization is proportional to the temperature, we expect that net alignment (which is responsible for the dipole moment density) should be inversely proportional to the temperature. Indeed, calculation based on statistical mechanics yield a dipole moment density,

$$
P = \frac{Np^2}{3k_BT}
\tag{4.23}
$$

where n is the molecule density, p is the magnitude of the permanent dipole moment per molecule, $k = 1.38 \times 10^{-23}$ J/K° is the Boltzmann constant, and T (K°) is the absolute temperature. In

liquids and solids, the molecular density is of order 10^{28} /m^3 being much larger than that of gases. Therefore, the permittivity of liquids and solids can be significantly larger than ϵ_0.

In most materials, electronic, ionic and molecular polarizations coexist. For example, the permittivity of water at room temperature (for static electric fields) is about $80\epsilon_0$. The molecular polarization accounts for only about 10 out of 80, and the rest is due to electronic and ionic polarizations of individual atoms. For theoretical prediction for dipole moments of molecules, quantum theoretical analysis is required.

4.4   Boundary Conditions for D and E

Maxwell's equations for static electric fields can be given either by

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\rho = \rho_{\text{free}} + \rho_{\text{bound}}) \tag{4.24}$$

$$\nabla \times \mathbf{E} = 0 \tag{4.25}$$

or by

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \tag{4.26}$$

as we have seen in Sec. 4.2. Integral forms of these equations are

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \quad (q = q_{\text{free}} + q_{\text{bound}}) \tag{4.27}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = q_{\text{free}} \tag{4.28}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{4.29}$$

where q is the charge enclosed by the closed surface, S. Using these integral forms, we can establish the boundary conditions for the fields $\mathbf{E}$ and $\mathbf{D}$ on a boundary surface between two dielectric media.

Let us consider Eq. (4.27) first. We assume an electric field $\mathbf{E}_1$ in the medium having a permittivity ϵ_1, and $\mathbf{E}_2$ in the medium having a permittivity ϵ_2, both near the boundary surface. Applying the surface integral to a pill box located at the boundary, we find

$$dS \ (E_{1n} - E_{2n}) = dS \ \Delta n \ \rho/\epsilon_0 \tag{4.30}$$

where E_{1n} and E_{2n} are the components normal to the boundary surface, Δn is the thickness of the pill box, and ρ is the charge density. As the thickness Δn approaches zero, it appears that the RHS of Eq. (4.30) vanishes, but it does not if a surface charge density σ (C/m^2) resides on the boundary. Therefore, we find

$$E_{1n} - E_{2n} = \frac{\sigma}{\epsilon_0} \quad (\sigma = \sigma_{\text{free}} + \sigma_{\text{bound}}) \tag{4.31}$$
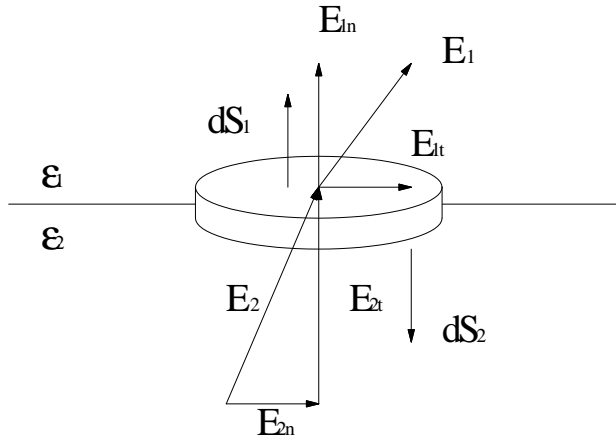
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Figure 4-4: Gauss' law applied to a pill box at a dielectric boundary.

Similarly, from Eq. (4.28), we find,

$$D_{1n} - D_{2n} = \sigma_{\text{free}} \tag{4.32}$$

where σ_{free} is the surface free charge density.

Although both Eqs. (4.31) and (4.32) are correct boundary conditions for the fields $\mathbf{E}$ and $\mathbf{D}$, Eq. (4.32) is much more useful in practical analyses because the bound surface charge σ_{bound} is not a directly measurable quantity. If instead the vector $\mathbf{D}$ is used, all the effects caused by the bound charge are incorporated in the permittivity, and only the free surface charge, σ_{free} appears in the boundary condition for $\mathbf{D}$. In particular, if there are no free surface charges on a boundary, the normal components of the displacement vector $\mathbf{D}$ must be continuous,

$$\mathbf{D}_{n1} = \mathbf{D}_{n2} \quad (\sigma_{\text{free}} = 0) \tag{4.33}$$

The vanishing line integral in Eq. (4.29) can be applied to a small rectangular loop placed on the boundary. Only the tangential components, E_{1t} and E_{2t}, contribute to the integral, and we find

$$dl(E_{1t} - E_{2t}) = 0,$$

or

$$E_{1t} = E_{2t} \tag{4.34}$$

This implies that the tangential components of the electric fields are always continuous at the boundary. (The continuity of the tangential component of the electric field holds quite generally even for time varying fields.)

As an example, let us analyze the fields in a capacitor partially filled with a dielectric slab as shown in Fig. **??**. For concreteness, we assume that the dielectric slab has a permittivity $\kappa\epsilon_0$,
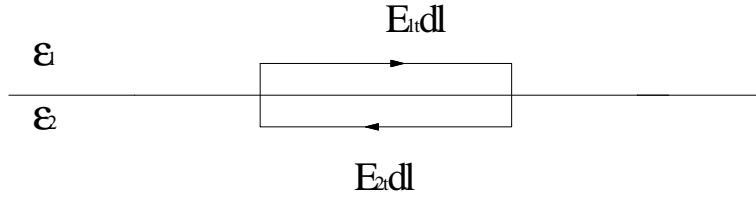
Figure 4-5: Line integral, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, applied to a loop at the boundary.

where κ is the relative permittivity. Normally, $\kappa > 1$. The electric field in the air regions is given by

$$E_a = \frac{\sigma}{\epsilon_0}$$

where σ is the free surface charge density residing on the capacitor plates. Since the dielectric slab carries no free charges, the continuity of the displacement vector

$$D_a = D_d \qquad (4.35)$$

applies. The displacement vectors in each region are

$$D_a = \epsilon_0 E_a$$

$$D_d = \kappa \epsilon_0 E_d$$

Then,

$$E_d = \frac{1}{\kappa} E_a \qquad (4.36)$$

and the field in the slab is indeed smaller than that in air as expected. The surface charge density induced on the dielectric slab is

$$\sigma' = \left(1 - \frac{1}{\kappa}\right)\sigma \qquad (4.37)$$

The potential difference between the capacitor plates is given by

$$V = (l - d)E_a + dE_d = \left[l + d\left(\frac{1}{\kappa} - 1\right)\right]\frac{\sigma}{\epsilon_0}$$

where l is the plate separation and d is the thickness of the slab. Assuming a plate area of A, we thus find a capacitance

$$C = \frac{A\epsilon_0}{l + d\left(\dfrac{1}{\kappa} - 1\right)} \quad \text{(F)} \qquad (4.38)$$
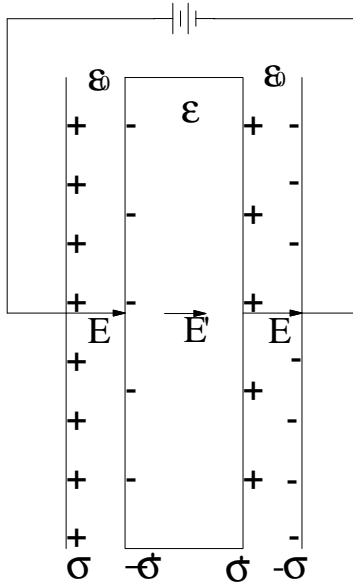
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Figure 4-6: Dielectric slab (permittivity ε) in a capacitor. The electric field in the slab is $E' = (\varepsilon_0/\varepsilon)E < E$.

which is consistent with the series equivalent capacitance of the two capacitors,

$$C_a = \frac{\epsilon_0 A}{l - d},$$

$$C_d = \frac{\kappa \epsilon_0 A}{d}.$$

Dielectric Sphere in an External Field

As a similar, but more complicated problem, we consider a dielectric sphere of radius a and permittivity ϵ placed in a uniform electric field $\mathbf{E}$ which is directed in the $+z$ direction. We assume that the sphere carries no free charges. The potential associated with the uniform electric field can be written in the spherical coordinates as,

$$\Phi_0 = -E_0 z = -E_0 r \cos \theta \tag{4.39}$$

The dielectric sphere perturbs both the external potential and electric field since surface charges are induced by the external field. Following the procedure developed in Sec. 2.3 for the spherical
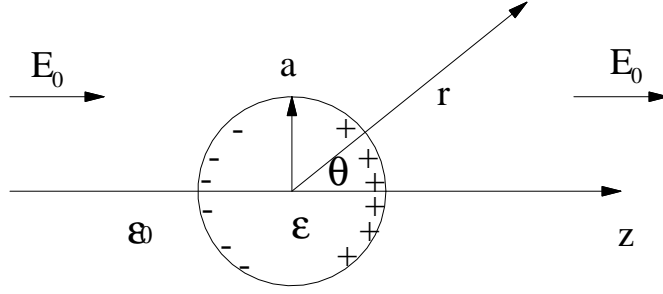
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Figure 4-7: Dielectric sphere (radius a, permittivity ε) in an external electric field E_0. Surface charges $+\sigma$ and $-\sigma$ are induced as shown and the perturbed potential is of electric dipole.

coodinates, we assume the total potential in the following form,

$$
\Phi(r,\theta) =
\begin{cases}
\Phi_0(r,\theta) + \sum_l A_l \left(\dfrac{r}{a}\right)^l P_l(\cos\theta) & r < a \\[4mm]
\Phi_0(r,\theta) + \sum_l A_l \left(\dfrac{a}{r}\right)^{l+1} P_l(\cos\theta) & r > a
\end{cases}
\tag{4.40}
$$

In this form, the continuity of the potential at the sphere surface $r = a$ is automatically satisfied. Note that the continuity in the potential is equivalent to the continuity in the tangential (E_θ in this case) electric field. The radial electric field can be found from

$$
E_r = -\frac{\partial \Phi}{\partial r}
$$

Performing calculations, we obtain the interior and exterior electric fields,

$$
E_{ri}(r,\theta) = E_0 \cos\theta - \sum_l A_l \frac{r^{l-1}}{a^l} P_l(\cos\theta) \quad r < a
\tag{4.41}
$$

$$
E_{re}(r,\theta) = E_0 \cos\theta - \sum_l A_l (l+1) \frac{a^{l+1}}{r^{l+2}} P_l(\cos\theta) \quad r > a
\tag{4.42}
$$

At the sphere surface, the normal components of the displacement vector must be continuous, that is

$$
\epsilon E_{ri}\big|_{r=a} = \epsilon_0 \, E_{re}\big|_{r=a}
$$

This yields

$$
\epsilon \left(E_0 \cos\theta - \sum_l A_l \frac{l}{a} P_l(\cos\theta) \right) = \epsilon_0 \left(E_0 \cos\theta + \sum_l A_l \frac{l+1}{a} P_l(\cos\theta) \right)
\tag{4.43}
$$

Since $\cos\theta = P_1(\cos\theta)$, only the $l = 1$ terms in the summations remain nonvanishing, and all higher order terms must vanish,

$$A_l = 0 \qquad (l \geq 2)$$

A_1 can be found from

$$\epsilon\left(E_0\cos\theta - A_1\frac{1}{a}\cos\theta\right) = \epsilon_0\left(E_0\cos\theta + \frac{2A_1}{a}\cos\theta\right)$$

or

$$A_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}aE_0 \tag{4.44}$$

and the final solution to the potential becomes

$$\Phi(r,\theta) = \begin{cases} -E_0 r\cos\theta + \dfrac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}E_0 r\cos\theta\ , & r < a \\[4mm] -E_0 r\cos\theta + \dfrac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}E_0\dfrac{a^3}{r^2}\cos\theta\ , & r > a \end{cases} \tag{4.45}$$

Note that outside the sphere, the potential due to the dielectric sphere (the second term) is a dipole potential. This is an expected result since on the sphere surface, equal but opposite surface charges are induced (Fig. **??**). The induced surface charge σ can be found from the discontinuity in the radial electric field

$$E_{re}|_{r=a} - E_{ri}|_{r=a} = \frac{\sigma}{\epsilon_0}$$

or

$$\sigma = \epsilon_0\frac{3(\epsilon - \epsilon_0)}{\epsilon + 2\epsilon_0}E_0\cos\theta \tag{4.46}$$

Let us examine a few special cases of the potential and electric field. When $\epsilon = \epsilon_0$, the perturbed potential vanishes. A sphere having the permittivity ϵ_0 has no effects on the applied potential. The induced surface charge vanishes. There are well expected results.

When $\epsilon \to \infty$, the potential becomes

$$\Phi(r,\theta) = \begin{cases} 0 & r < a \\[4mm] -E_0\cos\theta + E_0\dfrac{a^3}{r^2}\cos\theta & r > a \end{cases} \tag{4.47}$$

This corresponds to the potential due to a conducting sphere placed in an external electric field. Mathematically speaking, infinitely permittive dielectrics behave as if they were conductors, as long as the potential is concerned. However, such analogy must be used very carefully. (cf. Sec. 4.7)

When $\epsilon > \epsilon_0$, the electric field in the sphere is smaller than the unperturbed field,

$$\mathbf{E}_i = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}\mathbf{E}_0 \tag{4.48}$$

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However, the displacement vector is larger than $\epsilon_0 \mathbf{E}_0$,

$$\mathbf{D}_i = \frac{\epsilon}{\epsilon_0} \frac{3\epsilon_0}{\epsilon + 2\epsilon_0 \mathbf{D}_0} \tag{4.49}$$

Figure 4.10 shows the electric field lines and D vector lines for the case $\epsilon = 2\epsilon_0$. Note that D-lines are continuous (as they should be because we have assumed that the sphere carries no net free charges), but the electric field lines are discontinuous at the sphere surface.

In the limit $\varepsilon \gg \varepsilon_0$, the problem reduces to a conducting sphere placed in an electric field. The electric field in the sphere vanishes as expecetd.

Fields Due to a Permanently Polarized Sphere

Some materials (e.g. wax, barium titanate BaTiO$_3$) exhibit permanent polarization under certain circumstances. Let us assume a sphere of radius a having a uniform dipole moment density $\mathbf{p}$. Excess, unbalanced surface charge to appear on the sphere surface should have the same angular dependence as for the case of spherical dielectric placed in an external field. Therefore, outside the sphere, the potential is of dipole nature, and inside, we should have a uniform electric field directed in $-z$ direction. Therefore, we may assume

$$\Phi(r,\theta) = \begin{cases} A\dfrac{r}{a}\cos\theta & r < a \\[3mm] A\left(\dfrac{a}{r}\right)^2 \cos\theta & r > a \end{cases} \tag{4.50}$$

The radial electric fields inside and outside the sphere become

$$E_{ri} = -\frac{A}{a}\cos\theta \qquad r < a$$

$$E_{r0} = 2\frac{Aa^2}{r^3}\cos\theta \qquad r > a \tag{4.51}$$

Recalling the definition of the displacement vector

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.52}$$

and imposing the continuity of the radial components of $\mathbf{D}$ at the sphere surface, we obtain

$$-\frac{\epsilon_0 A}{a} + P = 2\frac{A}{a}\epsilon_0$$

or

$$A = \frac{1}{3\epsilon_0}Pa \tag{4.53}$$

Therefore, the electric field inside the sphere becomes

$$\mathbf{E}_1 = -\frac{1}{3}\frac{\mathbf{P}}{\epsilon_0} \tag{4.54}$$

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and the interior displacement vector is given by

$$\mathbf{D}_i = \frac{2}{3}\mathbf{P} \tag{4.55}$$

Note that the electric field and displacement vector are opposing each other, in contrast to the cases of induced polarization which we have analyzed earlier. A permanently polarized body (often called "electret" in analogy to magnet) is analogous to a permanent magnet which also exhibits antiparallel vectors, $\mathbf{H}$ and $\mathbf{B}$. In electrets, the permittivity ϵ is strongly nonlinear, just like the magnetic permeability of ferromagnetic materials.

4.5   Electric Energy

The physical meaning of the scalar potential Φ is the amount of work required to move a unit charge from one point to another, as explained in Chapter 2. Let us consider the amount of work required to construct a two-charge system with charges q_1 and q_2 separated by a distance r_{12}. The force to act between the charges separated a distance r is

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r^2}\mathbf{e}_r \tag{4.56}$$

Then, integrating this from $r = \infty$ down to $r = r_{12}$, we find the potential energy

$$\begin{aligned} U &= -\frac{q_1 q_2}{4\pi\epsilon_0}\int_\infty^{r_{12}} \frac{1}{r^2} dr \\ &= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r_{12}^2} \end{aligned} \tag{4.57}$$

This is the amount of work we have to do to bring the charge q_2 from $r = \infty$ to r_{12} from the charge q_1.

Note that the potential energy can be either positive or negative, depending on the sign of $q_1 q_2$. A positive potential energy is obviously an energy required to assemble a charge system. A negative potential energy must be interpreted as the amount of energy required to dismantle a charge system. For example, a hydrogen atom consists of a proton and an electron. Therefore, the potential energy of a hydrogen atom is negative, and to dismantle (or to ionize) the atom, a certain amount of energy must be given to the atom. (This energy for a hydrogen atom amounts to 13.6 eV = 2.2 $\times 10^{-18}$ J, which is not exactly equal to the potential energy

$$U = -\frac{e^2}{4\pi\epsilon_0 r} = -27.2 \text{ eV} \tag{4.58}$$

where $r_B \simeq 5 \times 10^{-11}$ m is the electron orbit radius (Bohr radius). The reason is the electron revolving around the proton has a positive kinetic energy, and the total energy involved in a

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hydrogen atom actually turns out to be

$$U + K = -\frac{e^2}{8\pi\epsilon_0 r} = -13.6 \text{ eV} \tag{4.59}$$

Derivation of this expression is left for an exercise.)

The potential energy found in Eq. (4.57) is equally shared by each charge, q_1 and q_2. That is, each charge acquires the same potential energy

$$U_1 = U_2 = \frac{q_1 q_2}{8\pi\epsilon_0 r_{12}} \tag{4.60}$$

when brought to a distance r_{12}.

For a system containing many charges, Eq. (4.60) can be generalized as

$$U = \sum_i \frac{1}{2} q_i \Phi_i \tag{4.61}$$

where Φ_1 is the potential at the location q_i due to all **other** charges. "Other charges" is emphasized here because the potential of a point charge at the location of the charge itself simply diverges. In the definition potential energy in Eq. (4.67), the "self energy" is thus excluded. For continuously distributed charges, the summation can be replaced by an integral,

$$U = \frac{1}{2} \int \Phi(\mathbf{r})\rho(\mathbf{r})dV \tag{4.62}$$

However, the Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4.63}$$

allows us to express the potential energy U in terms of the potential and electric field as

$$U = \frac{\epsilon_0}{2} \int \Phi \nabla \cdot \mathbf{E} \, dV \tag{4.64}$$

This may be rewritten as

$$\begin{aligned} U &= \frac{\epsilon_0}{2} \int [\nabla \cdot (\Phi\mathbf{E}) - \mathbf{E}\cdot\nabla\Phi] \, dV \\ &= \frac{\epsilon_0}{2} \oint_s \Phi\mathbf{E}\cdot d\mathbf{S} + \int \frac{\epsilon_0}{2} E^2 dV \end{aligned} \tag{4.65}$$

where the Gauss' theorem

$$\int \nabla \cdot \mathbf{A} \, dV = \oint \mathbf{A}\cdot d\mathbf{S}$$

and the relation

$$\mathbf{E} = -\nabla\Phi$$

have been used. The first surface integral vanishes since the integration is to be carried out at

$r = \infty$ where field and potential vanish. Therefore,

$$U = \int \frac{1}{2}\epsilon_0 E^2 dV \quad \text{(J)} \tag{4.66}$$

Note that this is positive definite, while the summation form in Eq. (4.66) can be either positive or negative. Where have we gone wrong? Recall the conditional statement following Eq. (4.61). In the summation form, the potential Φ_1 is to be understood as the one at charge q_i produced by all **other** charges. In other words, in the potential Φ_i in Eq. (4.61), contribution from the charge q_i itself is excluded because we have no way to calculate the potential right at a point charge. (It simply diverges for a point charge having no spatial spread.) On the other hand, the integral form, Eq. (4.66), is baded on the assumption that the electric field is calculable from a charge distribution through the Maxwell's equation. Once this assumption is made, then the integral form does contain the energy due to a point charge, although for an ideal point charge, it still diverges, that is, an infinitely large amount of energy (positive definite!) results for an ideal point charge. Later, we will see that even elementary particles (electron and proton) appear to have finite radii, and such divergence of potential energy may not occur in practice.

The potential energy given in Eq. (4.66) indicates that the quantity

$$\frac{1}{2}\epsilon_0 E^2 \quad \text{(J/m}^3\text{)} \tag{4.67}$$

should have the dimensions of energy density. Wherever an electric field is present, be it of electrostatic or dynamic origin, an energy density given in Eq. (4.67) is associated. It should be emphasized that Eq. (4.67) is a correct expression if the local electric field is known even in dielectrics. In dielectrics, however, the local (or microscopic) electric is a difficult quantity to evaluate precisely because each dipole certainly complicates the electric field. When we talk about an electric field in a dielectric, we mean an averaged (or smooth) electric field. Therefore, Eq. (4.67), although perfectly correct under any circumstances, is not very convenient, particularly when it is to be applied to dielectrics. A formula based on an averaged electric field would be more convenient. For this purpose, we replace the total charge density ρ with a free charge density in Eq. (4.62),

$$\frac{1}{2} \int \Phi \, \rho_{\text{free}} \, dV \tag{4.68}$$

Since ρ_{free} is the equantity that can be controlled by external means (e.g. power supplies), we may interpret Eq. (4.68) as the energy supplied to a system involving dielectrics. It may not be purely electric nature, for applying an external electric field may change mechanical and kinetic energy of the dielectric material. Recalling

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \tag{4.69}$$

and following the procedure developed for arriving at Eq. (4.66), we can rewrite Eq. (4.68) as

$$\frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} \, dV \tag{4.70}$$

where $\mathbf{E}$ and $\mathbf{D}$ are both understood to be the averaged electric field and displacement vector. Therefore, in a dielectric, the total energy density is given by

$$\frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{1}{2}\epsilon E^2 \tag{4.71}$$

Remember that this may contain energies of nonelectric nature (such as kinetic energy of electrons), as well as the electric energy.

4.6 Energy and Force in a Capacitor

As a concrete application of the energy expression, Eq. (4.70), we consider a capacitor filled with a material of a permittivity ϵ. When such a capacitor is charged by an external means (e.g. power supply), the energy stored can be evaluated from

$$U = \frac{1}{2}\int_v \mathbf{D}\cdot\mathbf{E}\,dV = \frac{1}{2}\int_v \epsilon E^2\,dV \tag{4.72}$$

where the volume V should cover all spatial regions wherever the fields, $\mathbf{E, D}$ are finite.

An alternative expression for the stored energy is

$$\frac{1}{2}CV^2 = \frac{q^2}{2C} \tag{4.73}$$

where V is the potential difference between the two electrodes, which is related to the charge q through

$$q = CV$$

Eq. (4.72) directly follows from the general formula

$$\frac{1}{2}\int \Phi\,\rho_{\text{free}}\,dV$$

for a capacitor, the volume integral should be replaced by a surface integral

$$\rho dV \rightarrow \sigma dS$$

since static charges can reside only on the surface of electrodes. Furthermore, the electrode surface is an equipotential surface. Then, the potential Φ can be taken out of the integral, and we have

$$\frac{1}{2}\Phi\int \sigma dS = \frac{1}{2}\Phi q$$

Replacing Φ with the potential difference V, we recover Eq. (4.73).

Equating Eq. (4.72) to Eq. (4.73), we obtain

$$\frac{1}{2}\int \epsilon E^2 dV = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

that is, the capacitance can be found from the energy stored in a capacitor.

In order to convince ourselves that the capacitance worked out this way is consistant with the one based on the potential calculation, we consider the coaxial cable. We have found in Sec. 3.5 that the capacitance per unit length of a coaxial capacitor of inner/out radii a/b and permittivity ϵ is given by

$$\frac{C}{l} = \frac{2\pi\epsilon}{\ln\left(\dfrac{b}{a}\right)} \quad \text{(F/m)}$$

To apply the energy method, we note the radial electroc field is

$$E_\rho = \frac{\lambda}{2\pi\epsilon}\frac{1}{\rho} \quad (a < \rho < b)$$

where λ is the line charge density on the inner electrode. Then, the energy stored in a unit length along the axis becomes

$$
\begin{aligned}
\frac{U}{l} &= \frac{1}{2}\int \epsilon E_\rho^2 2\pi\rho d\rho \\
&= \frac{\lambda^2}{4\pi\epsilon}\int_a^b \frac{d\rho}{\rho} \\
&= \frac{\lambda^2}{4\pi\epsilon}\ln\left(\frac{b}{a}\right) \quad \text{(J/m)}
\end{aligned}
$$

Equating this to

$$\frac{q^2}{2Cl} = \frac{\lambda^2 l}{2C} \quad \text{(J/m)}$$

we find

$$\frac{C}{l} = \frac{2\pi\epsilon}{\ln\left(\dfrac{b}{a}\right)} \quad \text{(F/m)} \tag{4.74}$$

which is indeed consistent with the familiar expression for the capacitance.

The capacitance of a given electrode configuration is in general determined by the geometrical factors of the electrodes. For example, the capacitance of the parallel plate capacitor is

$$C = \frac{\epsilon S}{l} \tag{4.75}$$

where S is the plate area and l is the plate separation distance. As we all know, unlike charges attract each other. Therefore, in a parallel plate capacitor, the top and bottom plates should attract each other. The plates themselves are subject to an expansion force, since like charges repel each other. Here, we wish to evaluate such internal forces in a charged capacitor. In designing a capacitor, such forces must be considered carefully to ensure mechanical strength.

Let us consider the parallel plate capacitor. We assume that the capacitor has been charged and disconnected from a power supply. Let us increase the separation distance by dl. In doing so, we have to do work because of the attracting force. The work required to increase the separation distance by dl is

$$dW = Fdl$$

where F is the yet unknown attracting force. The change in the separation distance causes a change in the capacitance by

$$\begin{aligned} dC &= \frac{\epsilon S}{l+dl} - \frac{\epsilon S}{l} \\ &\simeq -\frac{\epsilon S}{l^2}dl \end{aligned}$$

This in turn causes a change in the electric energy stored in the capacitor

$$\begin{aligned} dU &= \frac{1}{2}\frac{q^2}{C+dC} - \frac{1}{2}\frac{q^2}{C} \\ &= -\frac{1}{2}\frac{q^2}{C^2}dC \end{aligned}$$

Note that when $dl > 0$, $dC < 0$. Then, the stored energy increases. This increase in the stored energy should come from the work done, $dW = Fdl$. Then,

$$\begin{aligned} F &= -\frac{1}{2}\frac{q^2}{C^2}\frac{dC}{dl} \\ &= q^2\frac{d}{dl}\left(\frac{1}{C}\right) \end{aligned} \tag{4.76}$$

But the force is attracting and thus is opposite to the direction of dl. Therefore, the above result can be generalized as

$$F_l = -\frac{q^2}{2}\frac{d}{dl}\left(\frac{1}{C(l)}\right) \quad \text{(N)} \tag{4.77}$$

This method based on virtual work can be applied to any geometrical factors contained in an expression for a capacitance.

When a capacitor is kept connected to a power supply with a fixed potential (rather than a fixed charge as in the above example), the force should be calculated from

$$F_l = \frac{V^2}{2}\frac{d}{dl}C(l) \tag{4.78}$$

Either method should yield a consistent result.

When the capacitance of a parallel plate capacitor

$$C = \frac{\epsilon S}{l}$$

is substituted into Eq. (4.77) or Eq. (4.78), the force in each case becomes

$$F_l = -\frac{q^2}{2}\frac{1}{\epsilon S} \tag{4.79}$$

$$F_l = -\frac{V^2}{2}\frac{\epsilon S}{l^2} \tag{4.80}$$

However, since $q = CV = \epsilon SV/l$, these expressions are identical, although the dependence on the separation distance l is entirely different in both cases.

Example: Calculate the force to act between two, long, oppositely charged parallel lines of radii a and separation distance $D(\gg a)$.

The capacitance of this configuration has been calculated in Sec. 3.5, and given by

$$\frac{C}{l} = \frac{\pi\epsilon_0}{\ln\left(\dfrac{D}{a}\right)} \quad (D \gg a) \tag{4.81}$$

If the potential difference between the lines are fixed at V, we obtain from Eq. (4.78),

$$\begin{aligned}
\frac{F}{l} &= \frac{V^2}{2}\frac{d}{dD}\left(\frac{\pi\epsilon_0}{\ln\left(\dfrac{D}{a}\right)}\right) \\
&= -\frac{\pi\epsilon_0 V^2}{2}\frac{1}{D\left[\ln\left(\dfrac{D}{a}\right)\right]^2} \quad \text{(N/m)}
\end{aligned} \tag{4.82}$$

The negative sign indicates an attractive force as expected. The reader should numerically check that this force is not excessively large even for a typical high voltage power transmission line.

Let us go back to the parallel plate capacitor. The force to act on either plate is

$$F = \frac{q^2}{2}\frac{1}{\epsilon S} \quad \text{(N)} \tag{4.83}$$

Therefore, the force per unit area of the plate is

$$\frac{F}{S} = \frac{1}{2\epsilon}\left(\frac{q}{S}\right)^2 = \frac{\sigma^2}{2\epsilon} \tag{4.84}$$

where $\sigma(\text{C/m}^2)$ is the surface charge density. However, we have seen earlier that the electric field inside a parallel plate capacitor is given by
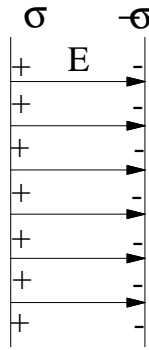
$$E = \frac{\sigma}{\epsilon} \tag{4.85}$$

Figure 4-8: The plates in a capacitor attract each other with a force per unit area, $\frac{F}{S} = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}$ N/m^2. The electric force in the direction of the electric field is a tensile force because the electric field lines tend to become shorter.

Therefore, the force per unit area becomes

$$\frac{F}{S} = \frac{1}{2}\epsilon E^2 \quad \text{N/m}^2 \tag{4.86}$$

Earlier, we have seen that the quantity

$$\frac{1}{2}\epsilon E^2 \tag{4.87}$$

has the dimensions of energy density, J/m^3. An alternative interpretation for this quantity is the force per unit area,

$$\frac{\text{J}}{\text{m}^3} = \frac{\text{N} \cdot \text{m}}{\text{m}^3} = \frac{\text{N}}{\text{m}^2}$$

One might prematurely interpret the force per unit as the pressure (N/m^2). However, this is not always correct. Depending on the direction of force relative to the direction of the electric field, the force appears as a tensile stress or as a pressure.

The attractive force in the case of the parallel plate capacitor is in the same direction as the electric field. The force per unit area

$$\frac{1}{2}\epsilon E^2$$

to act on the plate surface is directed from the region of zero energy density in the plate (recall that $\mathbf{E} = 0$ in a conductor) to the region of high energy density in the dielectric. This is in contrast to the concept of pressure, which acts from the higher to lower energy density region. We, therefore, conclude that the electric force in the direction of the electric field (either parallel or antiparallel) is of tensile stress nature.

On the other hand, the electric force perpendicular to the electric field can be interpreted as a pressure. To illustrate this, let us consider the boundary between the air and dielectric slab which is partially inserted into a parallel plate capacitor. The slab tends to be sucked in because the electric force always acts so as to increase the capacitor. If the gaps between the plates and the
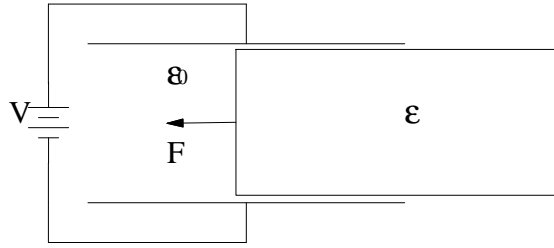
Figure 4-9: A dielecgtric slab partially filling a capacitor is pulled into the capacitor. The electric field is common $E = V/d$ and the force acts as a pressure from the higher energy density region $\frac{1}{2}\varepsilon E^2$ to the lower $\frac{1}{2}\varepsilon_0 E^2$.

dielectric slab are negligibly small, the electric fields in the air and slab are the same,

$$E = \frac{V}{d}$$

Therefore, the energy densities in the air and dielectric slab are

$$\frac{1}{2}\epsilon_0 E^2 \ , \ \frac{1}{2}\epsilon E^2$$

respectively. It is clear that the force perpendicular to the electric field is directed from the higher energy density region (slab) to the lower energy density region (air), and the difference,

$$\frac{1}{2}\epsilon E^2 - \frac{1}{2}\epsilon_0 E^2 \tag{4.88}$$

appears as a pressure.

The examples considered above are the special cases of a more general expression for electric stress which is in a tensor form. The force per unit volume to act on a charge density ρ is

$$\mathbf{f} = \rho \mathbf{E} \quad (\text{N/m}^3) \tag{4.89}$$

Since $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ (Maxwell's equation), the force becomes

$$\begin{aligned} \mathbf{f} &= \epsilon_0 (\nabla \cdot \mathbf{E})\mathbf{E} \\ &= \epsilon_0 \nabla \cdot (\mathbf{EE}) - \epsilon_0 (\mathbf{E} \cdot \nabla)\mathbf{E} \end{aligned} \tag{4.90}$$

where $\mathbf{EE}$ is a tensor defined by

$$\mathbf{EE} = \begin{pmatrix} E_x^2 & E_x E_y & E_x E_z \\ E_x E_y & E_y^2 & E_y E_z \\ E_x E_z & E_y E_z & E_z^2 \end{pmatrix} \tag{4.91}$$

Note that the divergence of a tensor is a vector. The last term in Eq. (4.90) is equal to

$$E\nabla E = \frac{1}{2}\nabla E^2 \tag{4.92}$$

for <u>static</u> electric field satisfying

$$\nabla \times \mathbf{E} = 0$$

To derive Eq. (4.92), we recall the general formula

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

Substituting $\mathbf{A} = \mathbf{B} = \mathbf{E}$, and noting $\nabla \times \mathbf{E} = 0$, we find

$$(\mathbf{E} \cdot \nabla)\mathbf{E} = \frac{1}{2}\nabla(\mathbf{E} \cdot \mathbf{E}) = \frac{1}{2}\nabla E^2 = E\nabla E$$

Physically, this equality means that for a curl-free fector, the curvature $(\mathbf{E} \cdot \nabla)\mathbf{E}$ and the gradient $E\nabla E$ should be identical. In other words, wherever the curl-free field is curved, there must be a gradient in the field's magnitude simultaneously associated with the curvature.

Substituting Eq. (4.92) into Eq. (4.90), we obtain

$$\mathbf{f} = \epsilon_0 \nabla \cdot (\mathbf{E}\mathbf{E}) - \frac{\epsilon_0}{2}\nabla E^2 \tag{4.93}$$

In the cartesian coordinates, this takes the following form

$$\mathbf{f} = \nabla \begin{bmatrix} \frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) & \epsilon_0 E_x E_y & \epsilon_0 E_x E_z \\ \epsilon_0 E_x E_y & \frac{\epsilon_0}{2}\left(E_y^2 - E_x^2 - E_z^2\right) & \epsilon_0 E_y E_z \\ \epsilon_0 E_x E_z & \epsilon_0 E_y E_z & \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) \end{bmatrix} \tag{4.94}$$

where

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}$$

For example, the x component of $\mathbf{f}$ is given by

$$f_x = \frac{\partial}{\partial x}\left[\frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right)\right] + \frac{\partial}{\partial y}\left(\epsilon_0 E_x E_y\right) + \frac{\partial}{\partial z}\left(\epsilon_0 E_x E_z\right) \tag{4.95}$$

In a dielectric, ϵ_0 should be replaced by ϵ, which may be spatially nonuniform.

Let us see if we can recover the two cases considered earlier. For simplicity, we assume that only the z component of the electric field is present. Then,

$$f_x = -\frac{\partial}{\partial x}\left(\frac{1}{2}\epsilon E_z^2\right)$$

$$f_y = -\frac{\partial}{\partial y}\left(\frac{1}{2}\epsilon E_z^2\right)$$

$$f_z = +\frac{\partial}{\partial z}\left(\frac{1}{2}\epsilon E_z^2\right)$$

These results clearly indicate that the electric stress force perpendicular to the field appears as a pressure, and the stress force in the same direction as the field appears as a tensile stress.

When we are interested in a total force to act on a dielectric body, we integrate the stress $\mathbf{f}$ over the volume of the dielectric to find

$$
\begin{aligned}
\mathbf{F} &= \int \mathbf{f}\, dV \\
&= \oint \left[\mathbf{E}(\mathbf{n}\cdot\mathbf{D}) - \frac{1}{2}(\mathbf{E}\cdot\mathbf{D})\mathbf{n} \right] dS
\end{aligned}
\tag{4.96}
$$

where $\mathbf{n} = d\mathbf{S}/dS$ is the unit normal vector on the closed surface S. Note that for a tensor $\mathbf{AB}$,

$$
\begin{aligned}
\int \boldsymbol{\nabla}\cdot(\mathbf{AB})dV &= \oint \mathbf{n}\cdot(\mathbf{AB})dS \\
&= \oint (\mathbf{n}\cdot\mathbf{A})\mathbf{B}dS
\end{aligned}
\tag{4.97}
$$

Example: A pararell plate capacitor with separation distance d is dipped vertically into a transformer oil which has a pemittivity ε and mass density ρ_m. If the capacitor is charged to a volatge V, how high does the oil in between the plates rise?

The electric field in between the plates is $E = V/d$ which is common to the air and oil. Therefore, the difference in the energy densities is given by

$$\frac{1}{2}(\epsilon - \epsilon_0)\left(\frac{V}{d}\right)^2 \tag{4.98}$$

Since the expected force is vertical, and thus perpendicular to the horizontal electric field, the energy difference acts as a pressure and is directed upward from the oil to the air. Therefore, the oil in between the plates will rise until the electric force is counterbalanced by the gravitational force. The gravitational pressure is given by

$$\rho_m g h \quad (\text{N/m}^2)$$

Equating this to the electric pressure, we obtain

$$\frac{1}{2}(\epsilon - \epsilon_0)\left(\frac{V}{d}\right)^2 = \rho_m g h$$

or

$$h = \frac{1}{2\rho_m g}(\epsilon - \epsilon_0)\left(\frac{V}{d}\right)^2 \tag{4.99}$$

When the oil permittivity is unknown, it can be determined from the height, h.