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Perceptual Geometry of Space and Form: Visual Perception of Natural Scenes And Their Virtual Representation

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Abstract. Perceptual geometry is an emerging field of interdisciplinary research whose objectives focus on study of geometry from the perspective of visual perception, and in turn, apply such geometric findings to the ecological study of vision. Perceptual geometry attempts to answer fundamental questions in perception of form and representation of space through synthesis of cognitive and biological theories of visual perception with geometric theories of the physical world. Perception of form and space are among fundamental problems in vision science. In recent cognitive and computational models of human perception, natural scenes are used systematically as preferred visual stimuli. Among key problems in perception of form and space, we have examined perception of geometry of natural surfaces and curves, e.g. as in the observer's environment. Besides a systematic mathematical foundation for a remarkably general framework, the advantages of the Gestalt theory of natural surfaces include a concrete computational approach to simulate or *"recreate images"* whose geometric invariants and quantities might be perceived and estimated by an observer. The latter is at the very foundation of understanding the nature of perception of space and form, and the (computer graphics) problem of rendering scenes to visually invoke virtual presence.

Introduction

In previous SPIE Vision Geometry conferences, we have introduced the concept of the Gestalt of a surface, and examined the role of perception in estimation of some geometric quantities of natural surfaces through the proposed multi-scale multi-resolution Gestalt theory of surfaces. Besides a systematic mathematical foundation for a remarkably general framework, the advantages of the Gestalt theory of natural surfaces include a concrete computational approach to simulate or *"recreate images"* whose geometric invariants and quantities might be perceived and estimated by an observer. The computational modeling of the Gestalt of surfaces is proposed within statistical learning theory. The model depends on the probabilistic functions whose measurements (for each observer) are within standard psychophysics. In this paper, we introduce the concept of perceptual space as a vehicle to develop the perceptual geometry of the visual space and object forms, discuss its significance in modeling human vision and related topics. In a nutshell, we propose that perception of space and form can be learned in computational input-output systems (that is, simplest forms of intelligent systems) that afford at least the following three capacities: (1) Memory to retain a representation. (2) Capability to perform and extract statistical correlations. (3) Ability to organize the results of correlation into a coherent response that fits into a feedback mechanism for further refinement. The relative simplicity of such basic learning systems is essential in extracting the most general principles that would underlie mechanisms for perception of space and form. Further, simpler mechanisms that are not biased by additional structures and mechanisms are more amenable to logical/algorithmic constructs, or even a rigorous mathematical formalism, the basic step towards computational implementation and design of experiments for (bio-behavioral) validation of the theory.

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and quantities might be perceived and estimated by an observer. The latter is at the very foundation of understanding the nature of perception of space and form, and the (computer graphics) problem of rendering scenes to visually invoke virtual presence. These applications justify a fundamental study of the theoretical foundations of perceptual geometry and attempts to develop computational models for its exemplars.

In this paper, we outline a new approach to computational modeling of perceptual space and visual perception of form. Our motivation comes from the concrete yet challenging problem of *digital communication of virtual perceptual presence within a network of computers by a group of intelligent agents* (including human users). Thus, a distant intelligent agent has access to digital data streams arriving through the network channels, potentially corrupted by some noise and subject to uncertainties of statistical type. The computational problem of virtual presence calls for digital encoding of the details of the distant scene so as to invoke the perception of presence by the human user on the receiving end of the communication channel. We propose to provide a computational model of the perceptual space and the environment surrounding the human subject. Then, encoding and broadcasting the components of the model would be the first step towards rendering the virtual space and object forms. Our theory lends itself to computational implementation by digital devices; however, it *is not* a mathematical theory *per se*, as it must incorporate human perception of the real world through physical measurements as well as complex biological computation of the environmental stimuli. Nonetheless, we wish to impart on the reader that a computational theory that agrees with experimental verification could be cast into a hybrid of rigorous mathematical components (that is, cast into the currently popular axiomatic framework) and experimental parts (also subject to verification of its agreement with observations, in their own right, much like theories in the physical sciences, up to prescribed approximate precision). Therefore, in exploring the criteria for computational modeling of space and form, the term “mathematical model” should be interpreted in this context.

The contents of the paper are briefly as follows. We begin our discussion by a brief historical account of attempts by notable thinkers in bridging between experience in physical space as a source of abstraction of concept of space and geometry. This topic is seen to be close to our point of view by comparing the empiricist theories regarding physical space and the role of learning and experience in forming the concept of the perceptual space, or its naive form, the concept of space as a “place” for objects. It is useful to compare and contrast the advances in physics of space-time with the complexity of generalizing them via the notion of metric to the perceptual space and perception of events. Examination of contributions by psychologists to estimate metric properties of the perceptual space points to logical shortcomings of their arguments similar to what are almost explicitly pointed out in articles by Helmholtz, Poincare’ and Einstein. The contradictions in metric theories of perceptual geometry point to a serious need for alternative theories. Our intent is to convince the reader of the role of *learning from experience* as the key behavioral requirement that must be incorporated in any theory of perceptual geometry. Therefore, the first section is primarily a review of a few key points from the historical articles by Helmholtz, and its impact on thinking of Poincare’ on the nature of geometry of perceptual space, and Einstein on the role of experience in theories of physical space-time, and his so-called “practical geometry.” At the same time, we provide a modern interpretation of their insights and arguments, and a brief mention of more modern follow-ups in the study of physical space-time. (For original source, cf. Geometrie und Erfahrung based on the invited address to Preussischen Akademie, Berlin, 1921, e.g. in Einstein’s Collected Works, or for an outline of his views, see P. A. Schilpp, ed., Albert Einstein, Philosopher-scientist, Tudor, New York, 1950, p. 355.) Cf. [1]-[13] for various aspects of the comments below. The references are partial to author’s modest background in the vast literature on space-time, and some are merely mentioned in view of the significance of the ideas of founders of such theories. More modern and often quantitatively supported accounts are available in monographs and text books, e.g. Palmer’s recent beautiful and comprehensive account [60] for topics related to perception, and the classic work of Green-Schwartz-Witten (Superstring Theory, Cambridge University Press, 1987) for the physical theory. For us, bridging the fruitful ideas from physics of space-time to perception is an important issue. Indeed, contrast and comparison with the physical theories are fundamental to our conviction, that the notion of Gestalt of surfaces (and its extension to Gestalt of places and events) is essential for success in establishing a geometric theory of visual perception. Further, we believe that attempts to formalize metric properties of perception of surfaces with full information from their texture, shading, etc. is likely to continue suffering from logical inconsistencies and limited applicability. Physical considerations and the history of developments of geometry of space-time from its differential topology, measurements of light propagation, and motion of free-falling bodies make it plane that the topology of the perceptual space as considered by contemporary cognitive scientists fails to satisfy the necessary separation properties that are necessary for existence of a metric geometry with desirable logical consistency.

In a following section, we review the theory of Gestalt of surfaces, itself inspired by the Gestalt school of psychology and ecological theorists in vision research [3][4][28][29][30]. This establishes *the second important link* between our theory and cognitive science of human perception, itself in harmony with experiments in biological vision and psychophysics verification. There have been numerous attempts to provide mathematical frameworks to model visual perception, for a selection cf. Helmholtz [1] and Poincare' [2] for the earliest critiques, and [11][12][17][18][20]-[26] for continuing progress in this direction, and [34][36][37][43][45][46][47]. In particular, Koenderink and van Doorn ([50] as one example of numerous publications) should be mentioned for substantial contributions to bridging between psychophysics and mathematics of perception. We have also briefly remarked on the mathematical formalism, as it is entirely new and quite distinct in spirit from the previous theories. Another reason to outline the mathematical concepts is their role in being used for computational and experimental study of Perceptual Geometry, with a brief discussion of their biological relevance.

Perception Of Place And Space

At first attempt, one is tempted to define the perceptual space is a medium in which perceptual geometry of natural surfaces, curves and objects is learned, whether by the human observer or any intelligent system capable of representing memories of stimuli in aggregates and inferring statistical correlates and probabilistic structures within them. The prerequisite to formation of the concept of perceptual space is that of physical space, where the observer is situated subject to experiences of perception of events and stimuli. After examination of the complexity in defining the notion of physical space, it becomes clear that one must settle an intermediate question before indulging in the more profound issues. Namely, to start from the concrete notion of "places" for objects and navigation of observers and attempt to generalize this notion in the context or perceptual organization, for instance, of vision.[2] We introduce the "naïve" concept of perceptual space as a vehicle to develop the perceptual geometry of the visual space and object forms. The naive perceptual space is an extension of the concept of a "place". We reserve the term "place" for concrete location of objects, where relative distances are estimated to judge how far or near objects are, the possibility of interaction or reaching for objects, navigation, etc.. Thus, we recognize a "place" as a matter of necessity for possibility of navigation and other tasks, where no abstraction is required beyond estimation of geometry of surfaces surrounding the observer, potentially with different strategies/mechanisms depending on the circumstances. Formation of the concept of "naive space," on the other hand, conveys a step beyond recognition of specific places for objects. The difference between the "naive perceptual space" and "place" is analogous to the difference between number "two" versus "two apples." The naive perceptual space, (provided that we accept its existence!) is an abstraction arising from the perception of "places." Formation of surfaces means, in general, the possibility of serving as a place for objects, or being unsuitable for such. Having the notion of the naive perceptual space implies that the observer is capable of decision arising from abstraction of relations. In a novel situation, a visual percept could serve as a place or not, even if the observer has not experienced the stimuli arising from that particular case, or without the need to have the entire information from visual stimuli. This is, of course, different from recognizing a particular image as a place that one has seen or has not.

We have preferred to use "naïve perceptual space" to emphasize the distinction from the more profound problem in cognitive science that bring under question even more basic problems, for instance, the necessity for formation of abstraction of physical space as a medium for cognition. Understanding the naïve perceptual space is a significant step in modeling human vision and related topics, such as image compression in the context of virtual reality and efficient realistic representation of spatial stimuli for distant users of teleconferences and other circumstances where realistic virtual presence is desirable. These applications justify a fundamental study of the theoretical foundations of perceptual geometry and attempts to develop computational models for its exemplars. On the other hand, understanding principles underlying formation of the cognitive perceptual space and insight into its continuity properties have profound implications for the human brain possessing a holistic medium as a prerequisite background for formation of perception of objects and events, and their relationships. Extending the analogy between "two" and "two apples" mentioned above, cognitive perceptual space compare to the naïve perceptual space as in "arithmetic" versus the number "two." Abusing the terminology for the sake of brevity, we drop the adjective "naïve" and limit our investigation to the first two instances in the hierarchy of spatial cognition and representation.

The questions of perception of space and mechanisms for spatial representation have received a great deal of attention by researchers in many areas and throughout centuries. For the purpose of this section, we wish to briefly review a small selection of attempts to understand spatial representation and perceptual geometry. Even if we were only interested in perception of surfaces, still we have to pay attention to the problems underlying perception of place and space, since perception of surfaces is influenced by the layout of the visual scene. Cf. [15][16][19][21][25][26][27]. Thus, perception of

[2] Although it is obligatory to consider all senses and diverse modes of experiencing stimuli that play a role in inference of "being in a place," we limit ourselves to visual perception, simply to avoid the complexity of fusion of senses.

surfaces in natural scenes is not independent of formation of the underlying "Gestalt" of perceptual space. The neural mechanisms for perception and recognition of "places" (e.g. recently unveiled in fMRI experiments by Nancy Kanwisher) indicate the biological relevance of this aspect, and potentially, in cognitive aspect through high-level neuronal processing and neuronal feedback mechanisms to other areas of the visual cortex.

The intellectual history of this area of investigation touches upon philosophy and metaphysics, art and anthropology, psychology, physics, and finally, the notion of abstract space in mathematics and theoretical physics. A great triumph of mathematics is to distill abstract notions of space and how diverse geometric structures are brought to strict logical scrutiny and investigation by axiomatic methods. Pure mathematics has advanced beyond discovery of non-Euclidean geometries and unification of space-time and gravity through special and general theories of relativity. Grothendieck's foundation of abstract algebraic geometry is by far the most versatile framework to formulate geometric questions of abstract spaces. On the other hand, many creators of abstract and physical theories of space have curiously paid attention to the challenge of investigating the notion of space endowed with structures imposed by the physical environment of human observers, and studied those properties in such 'perceptual spaces" arising from the human experience. Later we shall briefly mention several key observations by a small representatives of renowned proponents of such investigations, Leibnitz, Helmholtz, Poincare', Einstein and J.J. Gibson.

In cognitive science, on the other hand, a number of researchers have examined how humans and other animals represent space (Bloom et al., 1996; de Vega et al., 1996; Gallistel, 1990; Healy, 1998; and a number of articles in [57] for a cohesive account.) Study of spatial representation in non-humans is particularly enlightening in view of the necessity of establishing convincing evidence for biological basis of formation of some form of spatial representation. Beyond use of space itself for navigation and survival, such research suggests how spatial representations might influence reasoning, memory for nonspatial relations and problem solving. Cognitive scientists report variation in uses of spatial reasoning in specific cultural contexts, and even spatial representation as a basis for abstract thought. Evidence for the encoding of space ranges from psychophysics studies in humans, as well as behavioral studies in animals. As a result, behavioral researchers report observing a wide variety of internal mechanisms available for navigating through space. A body of research in this direction supports a theory of cognitive maps to find food and home. Recent findings, however, (e.g. Robert in [57]) pose some significant methodological challenges to showing that cognitive maps actually exist. Robert (cf. [57]) provide several alternative interpretations of existing data, and argues the basis for three very likely uses of space are encoding time, encoding number, and encoding order--and presents provocative analysis suggesting that rats use a spatial array to make transitive choices in a discrimination learning task. Despite extensive effort and a long intellectual history, past and recent studies uniformly indicate the complexity of the problem of formulating general, yet concrete hypotheses in perception of space and spatial representation. There has been some recent exciting progress in the direction of investigating the neurobiological substrates for perception of place and (naïve) space through functional magnetic resonance (fMRI) by Kanwisher et. al. (see above.) Yet, the same old question remains as illusive as ever: how does the brain perceive space and form of objects in it?

How much of the advances in the role of physics (experience) in determination of the structure and properties of space-time could be carried over to formulation of metric properties of the perceptual space? In vision science, there are numerous investigations ranging from attempts to estimate metric properties of the visual space in the style of Riemannian geometry to extensive examination of very subtle phenomena in visual perception of distance, size, constancy of shape, … Such results have shed light on numerous finer and subtler properties of perception of space and its geometric manifestations by human observers, yet without getting close to a definitive answer. References [15][21][32][38][43] discuss numerous partial successes as well as subtleties that arise in finding a compromise between the Riemannian version of geometry and the "metrics" for the perceptual space (as if the existence of a rigorous Riemannian metric is a given!) Dynamic of eye movements enters any direct geometric approach through complex nonlinearities [52], so the well-known texture-based theories of perception of surface geometry (e.g. [41][48][39][47]) also encounter mathematical complexities to account for subtleties and diversity of circumstances in of human visual perception. All such pitfalls, however, point to new directions to be followed in formulating the mathematics underlying perception of places and space. Learning theory is a first profound step in this direction (e.g. [40]), and understanding its implications in forming the appropriate mathematical framework for nonlinear complex biological/behavioral systems (e.g. [31][33][59][60][61][62].)

The study of intrinsic geometry of space, and space-time in physics has a long and glorious history, culminating in Einstein's theory of special and general relativity. Before the 20th century, however, empirical study of physical space is often influenced by consideration of human experience and perception of events, as the quoted passages from Helmholtz adequately illustrates. The following passage is selected from the 1876 paper of Herman von Helmholtz, "The Origin And

Meaning Of Geometric Axioms." Helmholtz argues that the axioms of geometry, "…taken by themselves out of all connection with mechanical propositions, represent no relations of real things… This is true, however, not only of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry… As soon as certain principles of mechanics are conjoined with the axioms of geometry, we obtain a system of propositions that has real import and which can be verified or overturned by empirical observations, as it can be inferred from experience. If such a system were to be taken as a transcendental form of intuition and thought, there must be assumed a pre-established harmony between form and reality…." Indeed, Helmholtz had the foresight that a future form of geometry based on the human experience should arise from the ashes of Riemannian geometry, the most general form of axiomatic geometry compatible with principles of classical mechanics and optics of the time. How does human experience, then, lead to discovery of the new geometry that takes into account principles of mechanics and the environment?"

"In conclusion, I would again urge that the axioms of geometry are not propositions pertaining only to the pure doctrine of space. As I said before, they are concerned with quantity. We can speak of quantities only when we know some way by which we can compare, divide, and measure them. All space measurements, and therefore all ideas of quantities applied to space, assume the possibility of figures moving without change of form or size. It is true that we are accustomed in geometry to call such figures purely geometric solids, surfaces, angles, and lines because we abstract these from all the other characteristics, physical and chemical, of natural bodies. Only one physical quality, rigidity, is retained. We have no mark of rigidity of bodies or figures other than congruence whenever they are superimposed on one another, at any time or place and after any revolution. We cannot, however, decide by pure geometry and without mechanical considerations whether the coinciding bodies may not both have varied in the same way."

"If it were useful for any purpose, we might with perfect consistency look upon the space in which we live as the apparent space behind a convex mirror with its shortened and contracted background. We might also consider a bounded sphere of our space, beyond the limits of which we perceive nothing, as infinite pseudospherical space. We should then, however, have to ascribe to the bodies, which appear as solid—and to our own bodies, at the same time….. We would also have to change our system of mechanical principles entirely, for even the proposition that every point in motion, if acted upon by no force, continues to move with unchanged velocity in a straight line is not adapted to the image of the world in the convex mirror. The path would indeed be straight, but the velocity would depend upon the place."

"Thus the axioms of geometry are concerned, not only with space relations, but also with the mechanical behavior of solid bodies in motion. The concept of a rigid geometric figure might indeed be conceived as transcendental in Kant's sense, that is, as formed independently of actual experience, which need not exactly correspond to it, any more than natural bodies ever in fact correspond exactly to the abstract conception we have obtained of them by induction. Taking the concept of rigidity thus as a mere ideal, a strict Kantian might look upon the geometric axioms as propositions given a priori by transcendental intuition, which no experience could either confirm or refute, because it must first be decided by them whether any natural bodies can be considered rigid. But then we should have to maintain that the axioms of geometry are not synthetic propositions, as Kant held them: they would merely define what qualities and behavior a body must have to be recognized as rigid. But if to the geometric axioms we add propositions relating to the mechanical properties of natural bodies—if only the axiom of inertia or the single proposition that the mechanical and physical properties of bodies and their mutual reactions are, other circumstances remaining the same, independent of place—such a system of propositions has a real import which can be confirmed or refuted by experience, but for the same reason can also be got by experience. The mechanical axiom just cited is, in fact, of the utmost importance for our whole system of mechanical and physical conceptions. That rigid solids, as we call them (they *are* really elastic solids of great resistance), retain the same form in every part of space if no external force affects them is a single case falling under the general principle."

"For the rest, I do not, of course, suppose that mankind first arrived at space intuitions in agreement with the axioms of Euclid by any carefully executed system of exact measurement. It was rather a succession of everyday experiences—especially the perception of the geometric similarity of great and small bodies, possible only in flat space—that led to the rejection as impossible of every geometric representation at variance with this fact. For this no knowledge of the necessary logical connection between the observed fact of geometric similarity and the axioms was needed, but only an intuitive apprehension of the typical relations among lines, planes, angles, etc., obtained by numerous, attentive observations—an intuition of the kind the artist possesses of the objects he is to represent and by means of which he decides surely and accurately whether a new combination which he tries corresponds to their nature. It is true that we have no word but *intuition* to mark this, but it is knowledge empirically gained by the aggregation and reinforcement of similar recurrent impressions in memory, not a transcendental form given before experience. That other such empirical intuitions of fixed typical relations, when not clearly comprehended, have frequently been taken by metaphysicians for a priori principles is a point on which I need not insist."

Toward the end of the nineteenth century, Poincaré [2] argued against the possibility to experimentally deciding which of the mutually exclusive Riemannian geometries of constant curvature (hyperbolic, elliptic, or flat Euclidean) applies to the physical space surrounding us. Poincaré discusses also the situation of perceptual space, and its relation to the physical space. It is interesting to note that the conceptual construction of the notion of space in modern physics is based on the empirical fact observed by Poincaré [2]. According to Poincaré, measurement is performed of empirically given physical objects in space, whether rigid bodies or light rays. Also, there exist two kinds of alteration of physical objects, changes of state and changes of position. In contrast to the former, it is the latter type of change that can be reversed by the arbitrary motions of our bodies. "That there are *bodily objects* to which we have to ascribe within a certain sphere of perception no alteration of state, but only alterations of position, is a fact of fundamental importance for *the formation of the concept of space* (in a certain degree even for the justification of the notion of the *bodily object* itself)." The important conclusion regarding the structure of space that Poincaré derives is that, experiment can tell us only of the relations that hold among physical objects, cf. [2]. The important conclusion emerging from argument of Poincare' is that experience can neither confirm nor refute a geometry, whichever geometry it may be. For Poincaré, one chooses geometry merely as a matter of convention. We select that system of geometry that enables us to formulate the laws of nature in the simplest way. Some three decades later, Einstein acknowledged the contributions of Helmholtz and Poincaré, discussed the role of experience in theories of physical space-time in his invited address titled Geometry and Experience to the Prussian Academy [3](Berlin, 1921.) Einstein introduced geometry as a form of theoretical physics, and coined the term "practical geometry," in contrast to Hilbert's abstract axiomatic geometry that he considered in the domain of mathematics. A *bodily object* is called "practically rigid" by Einstein. The position of either of two given practically rigid bodies can be changed without changing the position of the pair as such: So we get the concept of *"relative position,"* a special case of which is "contact" of two bodies at a point. Any two points on a practically rigid body define a "stretch." From this point on, one can follow the course of discovery of metric properties of the physical space based on the invariance of length of rigid bodies. Einstein's theory of general relativity showed that Poincare' was not completely right in his conclusion about the physical space. However, it remains to see how his view regarding the perceptual space holds with time. In modern theories of perception (e.g. [15]-[19]), one knows from numerous psychophysical experiments that the position of observer plays an important role in her/his perception of length and other metric properties in distances that relativistic changes are absent. The modern arguments against conclusion of Poincaré must still account for the subtleties of overcoming short-distance dependence on position of observer and his/her subjectivities, unlike the case of the physical space (see below.) The physical space contains the stimuli and observers, but it is transformed by the senses beyond our recognition! Which are the rules of perceptual transformations of space? If we start from the physical space, we must address the question of transformations of one form of space into another one.

The question of how much of the geometry of space and space-time could be derived from experiments is settled in a brilliant paper by Herman Weyl (Zur Infinitesimalgeometrie … Reprinted in Weyl's collected works Gesammelte Abhandlungen, Springer 1968, pp. 195-207). According to Weyl, the affine and the metric structures of space-time can be derived from experiments measuring propagation of light and the motion of free-falling bodies. More generally, Ehlers, Pirani and Schild provide a constructive realization of Weyl's program and accomplish to derive the conformal, projective and metric structure of space-time from measurements on the basis of light propagation and motion of free-falling bodies (see General Relativity: Papers in honor of J. L. Synge, Oxford University 1972.) The latter implies that measurement of length and time intervals are essentially derived operations in mathematics, based on measurements of fundamental physical phenomena. In Ehler's words " It has been shown on the basis of simple facts the space-time geometry of general relativity can be constructed without resorting to concepts or theorems of theories which presuppose such a geometry. … only concepts by which relations between events, particles, and light rays are describable have been introduced. This fully agrees with Leibnitz' position of viewing space and time not as objects, but rather as sets of spatial or temporal relations among things." (Cf. Philosophie und Physik, Wissenschaftverlag, Mannheim 1988, pp. 145-162, and also Coleman and Korte article "Jet bundles and path structures," Journal of Mathematical Physics (1980) Volume 21, pp. 3513-3526 for a correction of Ehler-pirani-Schild argument.)

Our intention is to investigate how the intermediate level (or surface-based) visual perception influences perception of the (naïve) space, and in turn, how spatial representation of space is learned from the optical information in the environment. One method of study is to propose the naïve perceptual space obtained from transformation of the physical space (places)

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[3] The original source (Geometrie und Erfahrung) can be found in Einstein's Collected Works, or alternatively, an outline of his views is paraphrased in P. A. Schilpp, ed., Albert Einstein, Philosopher-scientist (Tudor, New York, 1950, p. 355.)

defined by the environmental optics. Popular (image-based) theories in computational vision start with the two-dimensional images acquired by a camera or, presumably, the impression upon the surface of the retina. The central problems of vision propose algorithms for solving the inverse problems that recover the three-dimensional shape of objects in a scene from their two-dimensional images. In this formulation, the problem is ill posed for individual two-dimensional images, and it is necessary to invoke specific properties of the image (or the sequence of images, as in binocular vision or motion parallax) in order to derive additional constraints so as to transform the original questions into well-posed inverse problems. The optical properties of surfaces of objects in the scene, such as texture, color, and shading have been investigated in detail, and numerous algorithms propose to solve the above-mentioned central problem in vision. None of the algorithms so far have succeeded to solve a sufficiently general class of the problems for images in natural scenes. This is not surprising in view of the complexity of the visual cortex at any scale and from all angles of study, whether neurobiological or behavioral! In short, a naive theory of perception of "place," or the primitive notion of the space where the observer is located, follows from investigation of perceptual geometry of surfaces. As implied by Helmholtz discussion, perception of space (in the sense of cognitive science proper) requires consideration of more intricate factors. Regardless of types of mechanisms employed in perception of space, perception of surfaces are influenced by how the observer perceives spatial organization of objects and representation of events in her/his perceptual geometry of space. This factor, in turn, is incorporated in our perceptual geometry [31][33] as part of the subjective function evaluating the most probable choice for the Gestalt of surfaces. This is one important factor in favor of the Gestalt theory of surfaces, as computationally and experimentally the most accessible part of perception of form and space. It is also possible to formulate emergence of the naïve perceptual space as a result of categories of perceptual transformations of samples of observer's environments. An observer, therefore, learns to unravel these transformations, and to establish cause and effect relationships, expectations, navigation, rewards and punishments,… Perceptual transformations are observer-dependent, and a subjective function optimizes the probability of learning the optimal rules. The mechanism of learning takes place through incremental correlations and establishing an optimal statistics of the representations of stimuli and how they relate to the physical world around the observer.

Perceptual Geometry Of Surfaces

The mathematical concepts for solution of (restricted) vision problems cover a broad range. For example, the Gestalt and the ecological schools have been content with qualitative analysis and heuristic arguments (see Palmer for an excellent account, and). More recent theories are often supplemented by algorithmic and computational approaches, as pioneered by David Marr and continued by the following generations (see Palmer [59] for a comprehensive account in a modern context.) Statistical and learning theoretic approaches have opened fresh views of the vision problem, nonetheless, still with limited success (e.g. as in [40] for successful pioneering prototype.) There is also the remarkable trend in convergence of a vast body of experimental evidence from neurobiological studies, cognitive theories and psychophysical results in explaining various visual phenomena, from illusory contours to Fourier-like decomposition in spatial channels of visual stimuli (De Valois et. al. [49]). Of course, Fourier representation is merely a mathematical construct, and variants of wavelets could be also used in conjunction with learning theory and sparse representation theory, to provide alternative models [60][61]. Nonetheless, it is fair to say that there are still many problems in vision that cannot be solved with the existing computational tools and mathematical theories. Thus, it is natural to explore new ways to understand the fascinating process of vision, potentially with introduction of analytic concepts that might fall well beyond the conventional logic of mathematics and its relentless rigor. Such mathematical theories based non-Aristotelian logic and subject to approximate reasoning were anticipated by Helmholtz, Poincare', and in the case of physical space subject to experience, by Einstein, among others. Especially, such theories appear to have been inspired by exploratory studies and speculations on the nature of perceptual and physical space, and their roles in human perception of natural stimuli and events, and even questions of consciousness.

In the companion paper in this volume, we present simulation of results that apply the proposed learning theoretic approach to test the theory in concrete cases of visual perception of forms of natural surfaces with some degree of structural regularity such as the human face. To our knowledge, all the results are new, and they present a fresh look at the interpretation of geometry, from Helmholtz and Poincare' to contemporary views by Gibson and Koenderink. The success of the theory is expected within realistic expectation to be at least as good as present theories of visual perception, as the simulation results indicate consistent agreement with observations and approximation by the appropriate mathematical theories.

The space is perceived through experience with our surroundings, places of objects, and the range of possibilities in maneuvering within the environment. From the point of view of perception, object recognition is an important factor in our survival and functionality in the world. Through our interactions and experiences, we learn the properties of objects, estimate their places, and distinguish them from one another. We learn, in particular, that rigid objects occupy a constant volume, and their bounding "surfaces" keep their visual shapes and forms. Acknowledgement of properties and places of objects are learned through a combination of senses. Eventually, our visual perception of the external world relies on our ability to distinguish various pieces of surfaces. We integrate collections of surfaces into parts of an object, we fill any missing information by inference and other mechanisms that develop as part of our survival strategy. Thus, a theory of visual perception of surfaces is at the heart of any comprehensive theory of human perceptual organization, and in particular, any theory of space that advocates experience as an important factor.

The problems of figure-ground separation and scene segmentation in perceptual geometry could be formulated in terms of structural regularity of regions of images in statistical and information theoretic terms. Intuitively, as well as in psychophysical studies performed by cognitive scientists, perception of local structural regularity is fundamentally correlated with perception of local symmetry of surfaces, and under parallel projection of planar surfaces, with local symmetries of their images. In other words, such local symmetries distinguish prevalent regularity of common surfaces in the environment from randomness in arbitrary composition of colored dots; or what is the same, they distinguish between a meaningful image versus a generic pattern of a totally random selection of light intensities in matrices encoding local incoherence in optical properties. From a mathematical point of view, it can be shown that in the space of all possible patterns of light (i.e. all large matrices of same size with non-negative coefficients), the set of possible images of natural scenes is a very small subset. In the technical jargon of mathematical analysis, true images form a subset of measure zero in the space of all possible 2-dimensional patterns representing arbitrary light intensities.

Questions regarding the nature and mechanisms of human vision have attracted scientists for some time in diverse contexts. There are a number of different formulations and proposed theories in literature, addressing such problems. Below, we shall briefly highlight some of these findings related to the present research. The most notable recent contribution in this area is due to Gibson, advocating the structure of "optic array" at each point in the visible space as a key ingredient in his "ecological theory." Historically, Leonardo's theory of radiant pyramid objects can be considered as the origin of theories of visual perception similar to the ecological theory of Gibson that are based on the optical structure of points in space. Also, Keppler's final solution to the problem of visual image formation and the retinal image is an indispensable key in almost all visual theories until Gibson's ecological theory. However, Gibson rejected the claim that the retinal image is the starting point for visual processing. Gibson suggested replacing the classical approach to "depth" or "space" perception by an approach that emphasized the perception of surfaces in the environment. The environment consists of textured surfaces, which are themselves immersed in a medium (air). Gibson argues that we need an appropriate geometry to describe the environment, which will not be necessarily based on abstractions such as "points" and "planes." An ecological theory must take surfaces and texture elements as its starting point. To perceive things rather than nothing, light must be structured. In order to describe structure in light, we need an "ecological theory of optics." An ecological optics must cut across the boundaries between physical and physiological optics and the psychology of perception. Gibson argued that it is the total array of light beam reaching an observer, after structuring by surfaces and objects in the world, which provide direct information about the layout of these surfaces and objects, and about movement within the world and by the observer. Cf. [5]-[10]. These ideas clearly influence any empirical approach to design learning algorithms in perceptual geometry.

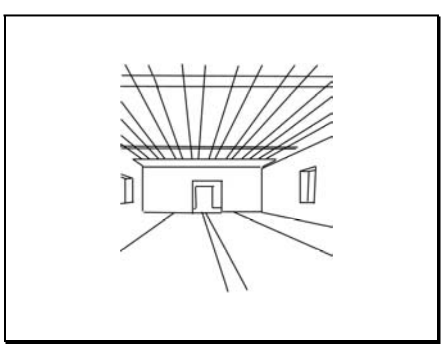
Learning The Geometry Of Forms And Places

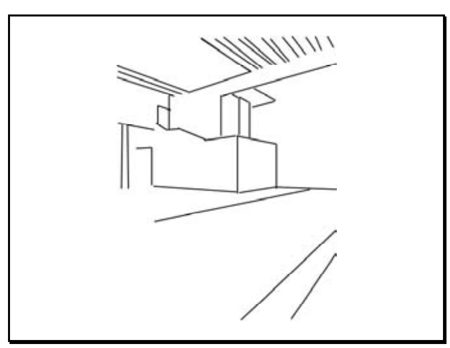
How do humans and other animals learn to represent space? The significance of this question for our topic lies in the empiricist approach adopted in our theory, along with evolution of concepts suggested in the preceding paragraphs. The process of learning a representation of space, and the nature and variability of representations are all fundamental topics of active research. The important consequence of such studies for us lies in taking the point of view of learning theory. Cf. [52]-[55]. A naive definition of the basic form of an intelligent system is as follows. An intelligent system in an input-output system that has at least the following capabilities: (a) Memory, that is, any form of retaining certain representation of inputs for the period of time during its exposure to its environment. (b) Internal processing of the representation (or data) in the form of extracting correlation among objects that are stored in the memory, and incrementing its memory by the results of the processing. (c) The capacity to organize the results of (b) into a coherent set of outputs that eventually exhibit a high correlation with the physical constraints in its environment, thus leading to a feedback mechanism through further input.

The learning theoretic approach that we have developed is inspired by the visual system: take sample stimuli from the environment, preprocesses the collection of feature data to simplify the task of statistical inference, and processes the filtered data through discovery of higher statistical correlation. The heuristic grounds for adapting these basic concepts are supported by behavioral, neuro-physiological and psychophysical studies in vision, e.g. [40][42][46][54][55] and others mentioned in the references below. In our previous works, we have elaborated on a number of issues that are parts of the learning theoretic implementation of forming the perception of places. If we take the naive definition of space as the volume of the environment surrounding the observer, and defined by the surfaces perceived by the observer, then forming the naive perception of space could be explained by generalization of the learning theoretic methods in [31][33][61][62], as we shall briefly outline below.

In our previous paper [62], we provided a detailed learning theoretic treatment of perceptual geometry of objects, focusing on the case of physical models of space curves. In particular, we have answered the question of perceptually distinguishing two possibly novel space curves based on observer's prior visual experience of physical models of curves in the environment. The prototype for "physical curves", just like branches of trees and other solid models, are volumetric representations that have relatively small cross section versus the elongated remaining dimension. The learning algorithms are applied to samples of "physical curves", simulating the observer's experience and retaining such representations to process the statistics of stimuli impressed upon her/him by such physical models of curves. The intelligent system abstractly models the observer's capacity to derive two statistically independent local functions of curves. In analogy with standard differential geometry, we call these functions curvature and torsion functions. In standard differential geometry, the *Fundamental Theorem of Space Curves* states that two differential invariants of a curve, namely curvature and torsion, determine its geometry. Our pair of invariants also distinguishes physical models of curves in the sense of perceptual geometry. If these functions are approximately the same for two space curves, then the observer confirms the two are the same. A perceptible difference in the pairs of invariants from two exemplars of physical curves, then, leads the observer to distinguish the two objects as dissimilar.

To learn the perceptual geometry of places (or the naive perceptual space), the observer must sample stimuli from the environment, distinguish surface orientation, slant, … from shading, texture, and other information. Next, from perspective information, create a representation of the three-dimensional space in her/his environment. The step involving estimates of geometry of surfaces has been treated in our previous papers [31][33][60][61]. Motion in space, eye movements, parallax, and other mechanisms provide sufficient sample of the environment stimuli so that the incremental correlations from stored representations could provide statistically least dependent invariants, from which, perceptual organization of surfaces could be inferred. One important and interesting case is inference of perspective dept in interior of buildings and other spaces with reasonably flat pieces of surfaces dominating the surrounding. Our preliminary results in this direction indicates that the solid geometry of space can be inferred through learning in the cases with simpler interior environments (where surface geometries can be estimated, as in the images below.) The result in the case of perspective depth is modeled by collecting a set of photographs from architectural elements, extracting the prominent edges in black and while, and reconstructing the basic projective geometric data for inference of depth from the configuration of lines. Once this simplified geometric inference is achieved, representation of surface planes and their spatial relationships become a standard problem in computational vision. Illumination, texture and other optic information allow one to estimate shape from such auxiliary data. In practice, however, the variety of textured curvilinear surface patches their occlusions and multitude of other geometric features in most natural visual scene poses a computationally daunting combinatorial problem. A satisfactory compromise presented in our research (Assadi, Eghbalnia, Diemer, Uchill, to appear) attempts to model interior space of some modern architectures, where almost all surfaces are flat, textures are simplified and illumination is greatly simplified or in diffused light. Two samples of the preprocessed data are provided below.





Mathematical Aspects

In this brief section, we outline a mathematical formalism that we use to lay out the grounds for experimental validation of our theory, as well as measurements necessary for verification of Gestalt of surfaces, the fundamental notion underlying perceptual geometry of space and form. The point of view that we take toward vision is generally called *Active and Exploratory Vision.* Perception of natural events is in general active and exploratory, and vision is no exception to this realistic rule. Saccades and other eye movements are necessary parts of visual perception. Therefore, any realistic theory of vision must take into account the dynamics created by the evolution of the processing in neuronal networks, including the dynamics of eye movements. Another fundamental phenomenon that one must take into account is that of *visual attention*, cf. [53]-[57], for example. Any reasonably helpful discussion of the experimental and theoretical studies of these topics will take many more pages. We refer the reader to a collection of references that are representative of some of these studies.

In the following, we wish to formalize the notion of a dynamical system that describes the sequence of neuronal processing in the brain of an observer due to the stimuli from the environment. The neuronal processing are the result of, e.g. the interaction of photons attributed to the sight of an object S with the discrete array of photoreceptors and the resulting activities in neuronal networks in retina, the thalamus and the cortical areas. Such a dynamical system is a "coherent system" that converges to description of the Gestalt of S. Therefore, a computational implementation of the theory forecasts the design of a learning system that could output a representation of the Gestalt of a surface. Further processing in the visual cortex and interaction with other areas, e.g. the frontal cortex, hippocampus, amygdala, etc. adds more cognitive and affective details, association with prior experience, and possibly action. Our theory calls for only the processing part ending in the Gestalt of S, leaving out further refinements that might be added. We must mention, however, that the theory does not exclude top-down influence of these higher levels of processing on the mechanism of formation of the Gestalt. Indeed, there are successful computational implementations of top-down processing for visual perception and object recognition that indicate the added complexity of processing lower-level visual areas (e.g. V1, V2) as a result of feedback among the visual areas and by thalamo-cortical connections. Our point of view is to hide all such intricate details in holistic stages of the processing, regarding them as transformation of information within neuronal networks, and abstractly represent them by the evolution in a dynamical system of which we expect to extract only the limited information that describes the simplest holistic outcome of the system, namely the Gestalt of the surface.

The perceptual geometry of space is far more complex, as the following remark shows. The computational notion of perceptual space interpreted according to the optics in the layout of a scene varies drastically from one view to another. However, our expectation from the notion of space as an abstraction of the notion of place should transcend the organization of Gestalts of surfaces defining the visual scene (optical component of the physical space surrounding the observer.) Thus, understanding the perceptual space requires, seemingly, a more complex mathematical construct from the collection of dynamical systems that aim at elucidating the Gestalt of a surface. In short, the notion of Gestalt of surfaces is potentially useful for rendering the virtual space (in the sense of computer graphics), and as such, placing an observer in a virtual optical scene in order to convey the percept of virtual presence in a distant site. We certainly do not claim that present level of complexity of oracle dynamical systems (cf. below) would lead to insights beyond such computational applications.

Consider an observer receiving light rays reflected from a surface or a collection of surfaces. Let us mark the time $t = 0$ for the initial instant that the light rays reach the retina. For the observer to reliably report visual perception of a surface S, a threshold time interval τ for exposure is required. By definition, the neuronal processes at in the observer's brain are called *perceptually coherent* with respect to S at time τ. All neuronal processes after τ are reliably reported as percept of an object S - possibly with more details. All such Neuronal processes after time t are called coherent with respect to S.

Hypothesis. There is a first moment τ at which a globally coherent percept is formed (as a result of neuronal processing which is the result of a discrete collection of events such as photons interacting with discrete array of photoreceptors). From time τ until the first threshold of recognition the precept remains coherent. Thus, there exists a processing stage where the following two axioms apply:
1) Perceptual Coherence.
2) Persistence of Coherence.

Definition. The coherent percept at time τ is called *the Gestalt of the surface S.* Note that further processing after time τ refines the information content of the Gestalt of S.

We can formalize the mathematical framework of our theory by introducing some standard tools from dynamical systems in Hilbert spaces. This choice of description is merely to help us conceptualize certain properties and their logical consequences more explicitly, as well as further development toward computational algorithms for numerical implementation. Functional analysis, in particular, the theory of Hilbert space operators is a highly versatile area of mathematics that has been developed to provide the rigorous foundations of quantum theory. Once an observer sees an object or a whole scene, the process of vision begins with the impression of photons of various wavelengths on the photoreceptors in the retina. It is a well-established result in vision science that (statistically) a single photon (a quantum unit of light) can potentially stimulate a photoreceptor. Thus, the process of vision starts with a discrete array of visual processes that evolves into the cortical information processing of visual scenes, and it leads to perception and eventually a description of an object.

The state of cortical processing at each instant of time is given as points in a Hilbert space H. Thus, the evolution of the process, therefore, provides an orbit of a dynamical system. We know very little about the details of points in such a Hilbert space, and even less about the type of dynamical systems that describe evolution of cortical information processing. Nonetheless, we could take a few steps further based on the nature of the problem at hand. Namely, we seek just enough information through a finite number of measurements and observations to be able to answer specific questions about such a dynamical system. If we assign coordinates to points of the Hilbert space using any prescribed representation, then our statement translates into computing some function of a finite number of the coordinates of points. The finiteness assumptions are essential in the realistic theory. The informed reader realizes, however, that intermediate stages of a mathematical model with finite input and finite output may well involve arguments regarding convergence or divergence of infinitesimal processes. We introduce a key concept that distinguishes our theory from others, namely, the formalism of the notion of "purpose" in intelligent complex biological systems with respect to performance of their tasks. A convenient mathematical notion that helps us formalize this admittedly non-rigorous concept is that of a projection operator in a Hilbert space. To avoid misunderstanding regarding various interpretations of "purpose", we use the terminology "*decision*" that fits artificial intelligence without ambiguity, and should, therefore, be useful in computational implementation of our theory. As observed in Neurophysiological studies of vision, e.g. [54], the role of visual attention in perceptual tasks fits well with the informal concept of "purpose" and its formal counterpart, namely "decision."

A Decision D is a collection of projection operators in the Hilbert space H. The description of an object S is optimal with respect to a Decision D if its representative vector **v** is not in the kernel of any projection operator **p** in D (that is, **p**(**v**) is nonzero.) Two vectors **u** and **v** are equivalent with respect to a decision D if for every p in D **p**(**u**) = **p**(**v**). Assuming that physical properties of an object causes states of cortical information processing that is represented as a vector **v** at time t, then v encodes the instantaneous state of the brain accounting for all the stimuli that could be afforded by the physical properties of the object in the scene. As time passes, one obtains a sequence of such vectors encoding all the information regarding the evolution of the process. Let us call a sequence of vectors indexed by time is called a *time-indexed dynamical system*. A dynamical system is *oracle* at time τ with respect to Decision D if every projection $\mathbf{p}(\mathbf{v}_t)$ is nonzero for all times **t** greater than τ. The condition of oracle is a substitute for the informal assertion that we wish to consider only those aspects of the theory for which we can account by computational implementation and scientific measurements. Another reason for this condition is to bring the study of such dynamical systems to the domain of "calculus of oracles" (forthcoming article.)

Experimental validation of our theory can be cast into a mathematical framework in which familiar mathematical objects must be evaluated. Moreover, the neuronal and psychophysical basis for this theory must be taken into account to validate the relevance of our theory to human visual perception. In forthcoming papers, we describe design of experiments that could lead to validation or shortcomings of our theory.

Conclusion

Surfaces and their optics define the most primitive notion of places for objects. Perception of place and motion are related, and both are influenced by environmental factors. Abstraction of multiple experiences of *"place"* leads to perception of *"space."* The *Gestalt of surfaces* is the most basic global representation in cortex that carries the minimal optical information needed to estimate the object's form. Later stages of processing results in incorporating additional optically encoded surface details (texture, shading, relatively small land-marks or salient features…). Processing evolution of the

Surface Gestalt due to higher-level neuronal processes represents the finer details. We have described a geometrical model for the perceptual space. We have shown that there are methods to compute these structures. Several bio-behavioral questions remain to be studied--for example, validation of the theory via psychophysics. A model for perception of geometry of objects and their dynamics follows from optics of textured surfaces based on Gestalt. More generally, the starting point for Perceptual Geometry is Gestalt of a geometric object, that is, as the simplest piecewise smooth structure that underlies more complex models of perception by the human observer. A naive concept of perception of "place" follows from perceptual organization of Gestalt of surfaces that define the optics of the place. The naïve concept of place is sufficient for computational implementation of virtual presence in computer graphics. However, generalization of the naïve notion of perception of places to *perception of space* (in the sense of cognitive science) is still an open problem.

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[4] Abbreviations: *AJP: American Journal of Psychology JEP: Journal of Experimental Psychology JEP:HP&P: Journal of Experimental Psychology: Human Perception and Performance JOSA: Journal of the Optical Society of America PB: Psychological Bulletin P&P: Perception Fr Psychophysics PR: Psychological Review VR: Vision Research*

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