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# Quantum Electrodynamics. I. A Covariant Formulation

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Attempts to avoid the divergence difficulties of quantum electrodynamics by mutilation of the theory have been uniformly unsuccessful. The lack of convergence does indicate that a revision of electrodynamic concepts at ultra-relativistic energies is indeed necessary, but no appreciable alteration of the theory for moderate relativistic energies can be tolerated. The elementary phenomena in which divergences occur, in consequence of virtual transitions involving particles with unlimited energy, are the polarization of the vacuum and the self-energy of the electron, effects which essentially express the interaction of the electromagnetic and matter fields with their own vacuum fluctuations. The basic result of these fluctuation interactions is to alter the constants characterizing the properties of the individual fields, and their mutual coupling, albeit by infinite factors. The question is naturally posed whether all divergences can be isolated in such unobservable renormalization factors; more specifically, we inquire whether quantum electrodynamics can account unambiguously for the recently observed deviations from the Dirac electron theory, without the introduction of fundamentally new concepts. This paper, the first in a series devoted to the above question, is occupied with the formulation of a completely covariant electrodynamics. Manifest covariance with respect to Lorentz and gauge transformations is essential in a divergent theory since the use of a particular reference system or gauge in the course of calculation can result in a loss of covariance in view of the ambiguities that may be the concomitant of infinities. It is remarked, in the first section, that the customary canonical commutation relations, which fail to exhibit the desired covariance since they refer to field variables at equal times and different points of space, can be put in covariant form by replacing the four-dimensional surface $t$=const. by a space-like surface. The latter is such that light signals cannot be propagated between any two points on the surface. In this manner, a formulation of quantum electrodynamics is constructed in the Heisenberg representation, which is obviously covariant in all its aspects. It is not entirely suitable, however, as a practical means of treating electrodynamic questions, since commutators of field quantities at points separated by a time-like interval can be constructed only by solving the equations of motion. This situation is to be contrasted with that of the Schrödinger representation, in which all operators refer to the same time, thus providing a distinct separation between kinematical and dynamical aspects. A formulation that retains the evident covariance of the Heisenberg representation, and yet offers something akin to the advantage of the Schrödinger representation can be based on the distinction between the properties of non-interacting fields, and the effects of coupling between fields. In the second section, we construct a canonical transformation that changes the field equations in the Heisenberg representation into those of non-interacting fields, and therefore describes the coupling between fields in terms of a varying state vector. It is then a simple matter to evaluate commutators of field quantities at arbitrary space-time points. One thus obtains an obviously covariant and practical form of quantum electrodynamics, expressed in a mixed Heisenberg-Schrödinger representation, which is called the interaction representation. The third section is devoted to a discussion of the covariant elimination of the longitudinal field, in which the customary distinction between longitudinal and transverse fields is replaced by a suitable covariant definition. The fourth section is concerned with the description of collision processes in terms of an invariant collision operator, which is the unitary operator that determines the over-all change in state of a system as the result of interaction. It is shown that the collision operator is simply related to the Hermitian reaction operator, for which a variational principle is constructed.

## INTRODUCTION

THE predictions of quantum electrodynamics concerning higher order perturbation effects have long been discredited in view of the divergent nature of the results. Several attempts[1] have been made to arbitrarily remove supposedly objectionable features of the theory—the so-called "subtraction physics." All such efforts have been fruitless; either failing in their avowed purpose, or lacking internal consistency. The unqualified success of quantum electrodynamics in applications involving the lowest order of perturbation theory indicates its essential validity for moderately relativistic particle energies. The objectionable aspects of quantum electrodynamics are encountered in virtual processes involving particles with ultra-relativistic energies. The two basic phenomena of this type

[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934); W. Heisenberg, Zeits. f. Physik **90**, 209 (1934); W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942).

[2] R. Serber, Phys. Rev. **49**, 545 (1936); H. A. Bethe and J. R. Oppenheimer, Phys. Rev. **70**, 451 (1946).

are the polarization of the vacuum and the self-energy of the electron.

The phrase "polarization of the vacuum" describes the modification of the properties of an electromagnetic field produced by its interaction with the charge fluctuations of the vacuum. In the language of perturbation theory, the phenomenon considered is the generation of charge and current in the vacuum through the virtual creation and annihilation of electron-positron pairs by the electromagnetic field. If the electromagnetic field is that of a light quantum, the vacuum polarization effects are equivalent to ascribing a proper mass to the photon. Previous calculations have yielded non-vanishing, divergent expressions for the light quantum proper mass. However, the latter quantity must be zero in a proper gauge invariant theory. The failure to obtain this result from a gauge invariant formulation can be ascribed only to a faulty application of the theory, rather than to an essential deficiency thereof. When the electromagnetic field is that of a given current distribution, one obtains a logarithmically divergent contribution to the vacuum polarization current which is everywhere proportional to the given distribution. This divergent result expresses the possibility, according to present theory, of creating electron-positron pairs with unlimited energy, a situation that presumably will be corrected in a more satisfactory theory. Thus the physically significant divergence arising from the vacuum polarization phenomenon occurs in a factor that alters the strength of all charges, a uniform renormalization that has no observable consequences other than the conflict with the empirical finiteness of charge.

The interaction between the electromagnetic field vacuum fluctuations and an electron, or more exactly, the electron-positron matter field, modifies the properties of the matter field and produces the self-energy of an electron. The mechanism here under discussion is commonly described as the virtual emission and absorption of a light quantum by an otherwise free electron, although an equally important effect is the partial suppression, via the exclusion principle, of the coupled vacuum fluctuations of the electromagnetic and matter fields. In a Lorentz invariant theory, self-energy effects for a free electron can only result in the addition of an electromagnetic proper mass to the electron's mechanical proper mass. Calculations performed for a stationary electron[3] have yielded a logarithmically divergent electromagnetic proper mass, a divergence that results from the possibility of emitting light quanta with unlimited energy. It is here, as in the vacuum polarization problem, that modifications will be introduced in a more satisfactory theory. However, the electromagnetic proper mass merely produces a renormalization of the electron mass that has no observable consequences, other than the conflict with the empirical finiteness of mass.

It is evident that these two phenomena are quite analogous and essentially describe the interaction of each field with the vacuum fluctuations of the other field. The effect of these fluctuation interactions is simply to alter the fundamental constants $e$ and $m$, although by logarithmically divergent factors. However, it may be argued that a future modification of the theory, inhibiting the virtual creation of particles that possess energies many orders of magnitude in excess of $mc^2$, will ascribe a value to these logarithmic factors not vastly different from unity. The charge and mass renormalization factors will then differ only slightly from unity, as befits a perturbation theory, in consequence of the small coupling constant for the matter and electromagnetic fields,

$$e^2/4\pi\hbar c = 1/137.$$

We may now ask the fundamental question: Are all the physically significant divergences of the present theory contained in the charge and mass renormalization factors? Will the consideration of interactions more complicated than these simple vacuum fluctuation effects introduce new divergences; or will all further phenomena involve only moderate relativistic energies, and thus be comparatively insensitive to the high energy modifications that are presumably to be introduced in a more satisfactory theory? This series of papers represents an attempt to supply at least a partial answer to the question, which has acquired an immediate importance in view of recent conclusive evidence

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that the electromagnetic properties of the electron are not fully described by the Dirac wave equation. Fine structure measurements on hydrogen, deuterium,[4] and ionized helium[5] have revealed energy level displacements that imply the existence of a weak, short range repulsive interaction between electron and proton. Experiments on the hyperfine structure of hydrogen and deuterium,[6] together with electron $g$ value determinations for several states of gallium and sodium,[7] prove that the electron possesses a small additional spin magnetic moment.

Immediately upon completion of the Lamb-Retherford experiment, it was generally recognized[8] that the most probable explanation was to be found in higher order electrodynamic effects; the radiative corrections to the properties of a bound electron other than mass and charge renormalization. A provisional non-relativistic calculation[9] lent support to this view. However, it required a completely relativistic treatment[10] to demonstrate that radiative corrections could account simultaneously for the two apparently unrelated deviations from the Dirac electron theory. It is our major task to enlarge on this development.

In order to isolate the divergent aspects of quantum electrodynamics in a manner that is Lorentz and gauge invariant, it is necessary to employ a formulation of the theory that preserves these covariant features at all stages. The use of a particular reference system or gauge in the course of calculation can result in a loss of covariance in view of the ambiguities that may arise in a divergent theory. The first paper is occupied with the development of a suitable covariant formulation. In the second paper we treat the problems of electron and photon self-energy, together with the polarization of the vacuum. The third paper is concerned with the major topic, the determination of the radiative corrections to the properties of an electron, and the comparison with experiment. Scalar and vector matter fields will be discussed in a fourth paper. It is hoped that successive papers of this series will deal with such subjects as the corrections to the Klein-Nishina formula, the scattering of light by light, and by a Coulomb field.

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[10] J. Schwinger, Phys. Rev. **73**, 415 (1948).

## 1. COVARIANCE IN THE HEISENBERG REPRESENTATION

In this section, we employ the following notation: Greek subscripts assume values ranging from 1 to 4, and a repeated index is to be so summed. The coordinate vector of a four dimensional point $x$ is denoted by $x_\mu=(\mathbf{r},ict)$. The real time coordinate $x_0=(1/i)x_4=ct$ is also used. In particular, the four dimensional element of volume is defined as $d\omega=dx_0dx_1dx_2dx_3$. The four-vector potential of the electromagnetic field is $A_\mu(x)=(\mathbf{A}(\mathbf{r},t),\,i\phi(\mathbf{r},t))$, while $\psi_\alpha(x)$ designates the four-component Dirac spinor. The spinor index will often be suppressed; thus, if $A$ and $A^T$ designate a four-rowed matrix and its transposed matrix, $A\psi=\psi A^T$ is a four component spinor of which the $\alpha$ component is $A_{\alpha\beta}\psi_\beta=\psi_\beta A_{\beta\alpha}{}^T$. Similarly, the scalar product of two spinors, $\chi$ and $\psi$, is denoted by $\chi\psi=\chi_\alpha\psi_\alpha$. The notation $\gamma_\mu$ is used for the four Hermitian matrices that obey the anticommutation relations

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}. \tag{1.1}$$

The adjoint spinor $\bar{\psi}_\alpha(x)$ is defined by

$$\bar{\psi}(x)=\psi^+(x)\gamma_4=\gamma_4{}^T\psi^+(x), \tag{1.2}$$

where $\psi_\alpha{}^+(x)$ is the Hermitian conjugate of $\psi_\alpha(x)$. The so-called charge conjugate spinor $\psi_\alpha{}'(x)$ and its adjoint $\bar{\psi}_\alpha{}'(x)$ are represented by

$$\psi'(x)=C\bar{\psi}(x),\quad \bar{\psi}'(x)=C^{-1}\psi(x). \tag{1.3}$$

Here $C$ is a matrix such that

$$\gamma_\mu{}^T=\gamma_\mu{}^*=-C^{-1}\gamma_\mu C, \tag{1.4}$$

which has the property of being skew-symmetric:

$$C^T=-C, \tag{1.5}$$

and unitary:

$$C^+C=1. \tag{1.6}$$

In the latter equation, $C^+=C^{T*}$ is the Hermitian conjugate matrix. For the particular representation in which all elements of $\gamma_4$ are imaginary, while all elements of the other matrices are real, the conditions on $C$ are satisfied with $C=-\gamma_4$. With this choice, $\bar{\psi}'(x)=\psi^+(x)$; charge and Hermitian conjugation are equivalent. Finally,

$$\kappa_0=m_0c/\hbar, \tag{1.7}$$

where $m_0$ is the mechanical proper mass of the electron.

The equations of motion of the coupled electromagnetic and electron-positron matter fields can be derived from the variational principle:

$$\delta\int \mathfrak{L}d\omega=0, \tag{1.8}$$

where the Lagrangian density $\mathfrak{L}$ is

$$\begin{aligned}\mathfrak{L}=&-\frac{1}{2}\frac{\partial A_\mu(x)}{\partial x_\nu}\frac{\partial A_\mu(x)}{\partial x_\nu}\\ &-\frac{\hbar c}{2}\bar{\psi}(x)\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu(x)\right)+\kappa_0\right]\psi(x)\\ &-\frac{\hbar c}{2}\bar{\psi}'(x)\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu(x)\right)+\kappa_0\right]\psi'(x),\end{aligned} \tag{1.9}$$

and is so constructed that it is invariant with respect to Lorentz transformations, gauge transformations and charge conjugation. The proof of Lorentz invariance follows the conventional treatment and need not be repeated. Gauge invariance, that is, invariance under the combined transformations

$$\begin{aligned}&A_\mu(x)\rightarrow A_\mu(x)-\frac{\partial\Lambda(x)}{\partial x_\mu}\\ &\psi(x)\rightarrow\exp\left[-\frac{ie}{\hbar c}\Lambda(x)\right]\psi(x)\\ &\psi'(x)\rightarrow\exp\left[\frac{ie}{\hbar c}\Lambda(x)\right]\psi'(x)\end{aligned} \tag{1.10}$$

induced by a scalar function of position, $\Lambda(x)$, would be generally valid were it not for the term in the Lagrangian density that refers to the electromagnetic field alone. The addition to $\mathfrak{L}$ arising therefrom is

$$-\frac{\partial}{\partial x_\mu}\left[\left(A_\nu+\frac{1}{2}\frac{\partial\Lambda}{\partial x_\nu}\right)\frac{\partial^2\Lambda}{\partial x_\mu\partial x_\nu}\right]+\left(A_\nu+\frac{1}{2}\frac{\partial\Lambda}{\partial x_\nu}\right)\frac{\partial}{\partial x_\nu}\frac{\partial^2\Lambda}{\partial x_\mu{}^2},$$

of which the first term has no effect on the equations of motion. Hence gauge invariance is restricted to the group of generating functions that obey

$$\frac{\partial^2\Lambda(x)}{\partial x_\mu{}^2}=\Box^2\Lambda(x)=0. \tag{1.11}$$

Invariance under charge conjugation expresses the complete symmetry between positive and negative charge. The interchange of $\psi(x)$ and $\psi'(x)$, together with $+e$ and $-e$, evidently leaves the Lagrangian density unaltered.

In order to obtain the equations of motion for the matter field, it is necessary to express the Lagrangian density entirely in terms of $\psi(x)$ and $\bar{\psi}(x)$, or alternatively, $\psi'(x)$ and $\bar{\psi}'(x)$. By virtue of Eqs. (1.3), (1.4), and (1.5), the following relations hold

$$\begin{aligned}\bar{\psi}'\gamma_\mu\psi'&=\psi C^{-1T}\gamma_\mu C\bar{\psi}=\psi\gamma_\mu{}^T\bar{\psi}\\ \bar{\psi}'\psi'&=\psi C^{-1T}C\bar{\psi}=-\psi\bar{\psi},\end{aligned} \tag{1.12}$$

and therefore the third term of (1.9) can be written

$$-\frac{\hbar c}{2}\psi(x)\left[\gamma_\mu{}^T\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu(x)\right)-\kappa_0\right]\bar{\psi}(x).$$

We find, as the result of variation, apart from discarded divergences,

$$\begin{aligned}\delta\mathfrak{L}=&\frac{1}{2}\delta A_\mu\left[\Box^2A_\mu+\frac{1}{c}j_\mu\right]+\frac{1}{2}\left[\Box^2A_\mu+\frac{1}{c}j_\mu\right]\delta A_\mu\\ &-\frac{\hbar c}{2}\delta\bar{\psi}\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)+\kappa_0\right]\psi\\ &+\frac{\hbar c}{2}\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)+\kappa_0\right]\psi\delta\bar{\psi}\\ &-\frac{\hbar c}{2}\delta\psi\left[\gamma_\mu{}^T\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)-\kappa_0\right]\bar{\psi}\\ &+\frac{\hbar c}{2}\left[\gamma_\mu{}^T\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)-\kappa_0\right]\bar{\psi}\delta\psi=0,\end{aligned} \tag{1.13}$$

where

$$j_\mu(x) = \frac{iec}{2}[\bar{\psi}(x)\gamma_\mu\psi(x) - \bar{\psi}'(x)\gamma_\mu\psi'(x)] \quad (1.14)$$

represents the four-vector of charge and current; $j_\mu = (\mathbf{j}, ic\rho)$. It is consistent with the form of the commutation relations imposed on the field quantities to infer that

$$\Box^2 A_\mu(x) = -\frac{1}{c}j_\mu(x), \quad (1.15)$$

and

$$\begin{aligned} &\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu(x)\right) + \kappa_0\right]\psi(x) = 0 \\ &\left[\gamma_\mu{}^T\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu(x)\right) - \kappa_0\right]\bar{\psi}(x) = 0. \end{aligned} \quad (1.16)$$

The Dirac equations for the matter field can also be cast in the charge conjugate form

$$\begin{aligned} &\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu(x)\right) + \kappa_0\right]\psi'(x) = 0 \\ &\left[\gamma_\mu{}^T\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu(x)\right) - \kappa_0\right]\bar{\psi}'(x) = 0. \end{aligned} \quad (1.17)$$

To the equations of motion must be added a supplementary condition, and the commutation relations. The supplementary condition

$$\frac{\partial A_\mu(x)}{\partial x_\mu}\Phi = 0 \quad (1.18)$$

restricts the admissible states of the system, as characterized by the constant vector $\Phi$ of our Heisenberg representation. The compatability of (1.18) with the equations of motion is a consequence of the charge conservation equation

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0. \quad (1.19)$$

The customary Maxwell equations, involving the field strengths

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \quad (1.20)$$

rather than the potentials, appear as derived supplementary conditions:

$$\left[\frac{\partial F_{\mu\nu}(x)}{\partial x_\mu} + \frac{1}{c}j_\nu(x)\right]\Phi = 0. \quad (1.21)$$

The commutation relations, in their conventional canonical form, read

$$\left[A_\mu(\mathbf{r},t), -\frac{1}{c}\frac{\partial}{\partial t}A_\nu(\mathbf{r}',t)\right] = i\hbar c\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}') \quad (1.22a)$$

$$\{\psi_\alpha(\mathbf{r},t), (\bar{\psi}(\mathbf{r}',t)\gamma_4)_\beta\} = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'), \quad (1.22b)$$

where the bracket symbols signify the commutator and anticommutator, respectively:

$$[A,B] = AB - BA, \quad \{A,B\} = AB + BA. \quad (1.23)$$

We have written the non-vanishing brackets; the bracket symbols whose values are zero are:

$$[A_\mu(\mathbf{r},t), A_\nu(\mathbf{r}',t)], \quad \left[\frac{\partial}{\partial t}A_\mu(\mathbf{r},t), \frac{\partial}{\partial t}A_\nu(\mathbf{r}',t)\right],$$

$$\{\psi_\alpha(\mathbf{r},t), \psi_\beta(\mathbf{r}',t)\}, \quad \{\bar{\psi}_\alpha(\mathbf{r},t), \bar{\psi}_\beta(\mathbf{r}',t)\},$$

and, of course

$$[A_\mu(\mathbf{r},t), \psi_\alpha(\mathbf{r}',t)], \text{ etc.}$$

It should be noted that the particle field commutation relations are invariant with regard to charge conjugation. Thus,

$$\begin{aligned} &\{\psi_\alpha'(\mathbf{r},t), (\bar{\psi}'(\mathbf{r}',t)\gamma_4)_\beta\} \\ &= -\{(C\gamma_4{}^T\bar{\psi}(\mathbf{r},t)\gamma_4)_\alpha, (\psi(\mathbf{r}',t)C^{-1}\gamma_4)_\beta\} \\ &= (\gamma_4 C)_{\alpha\gamma}\{(\bar{\psi}(\mathbf{r},t)\gamma_4)_\gamma, \psi_\delta(\mathbf{r}',t)\}(C^{-1}\gamma_4)_{\delta\beta} \\ &= \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \quad (1.24)$$

A further remark concerns the consistency of the supplementary condition and the commutation relations. Since (1.18) contains the arbitrary point $x$, one will obtain additional supplementary conditions by commutation, unless

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}, \frac{\partial A_\nu(x')}{\partial x_\nu'}\right] = 0 \quad (1.25)$$

for arbitrary $x$ and $x'$. In actuality, the canonical commutation relations are such as to yield (1.25). It must be realized that the commutator, considered as a function of $x$, obeys the wave equation, whence the validity of (1.25) is assured provided the commutator and its time derivative vanish for $t = t'$. This is easily verified.

The physical quantities characterizing the distribution of energy and momentum in the field are combined in the canonical energy-momentum tensor

$$T_{\mu\nu}=\frac{1}{2}\left[\frac{\partial A_\lambda}{\partial x_\mu}\frac{\partial A_\lambda}{\partial x_\nu}+\frac{\partial A_\lambda}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\mu}-\delta_{\mu\nu}\left(\frac{\partial A_\lambda}{\partial x_\sigma}\right)^2\right]+\frac{\hbar c}{2}\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\nu}+\frac{\hbar c}{2}\bar{\psi}'\gamma_\mu\frac{\partial\psi'}{\partial x_\nu} \quad (1.26)$$

which satisfies the conservation equation

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu}=0 \quad (1.27)$$

since

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu}=-\frac{1}{c}j_\mu\frac{\partial A_\mu}{\partial x_\nu}+\frac{ie}{2}[\bar{\psi}\gamma_\mu\psi-\bar{\psi}'\gamma_\mu\psi']\frac{\partial A_\mu}{\partial x_\nu}=0. \quad (1.28)$$

The canonical tensor can be replaced by a symmetrical energy-momentum tensor

$$\begin{aligned}\Theta_{\mu\nu}=&\tfrac{1}{2}[F_{\mu\lambda}F_{\nu\lambda}+F_{\nu\lambda}F_{\mu\lambda}-\delta_{\mu\nu}\tfrac{1}{2}F_{\lambda\sigma}{}^2]\\&+\frac{\hbar c}{2}\left[\bar{\psi}\gamma_\mu\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)\psi+\bar{\psi}\gamma_\nu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu\right)\psi\right.\\&+\bar{\psi}'\gamma_\mu\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}A_\nu\right)\psi'\\&\left.+\bar{\psi}'\gamma_\nu\left(\frac{\partial}{\partial x_\mu}+\frac{ie}{\hbar c}A_\mu\right)\psi'\right]. \quad (1.29)\end{aligned}$$

However, it is only the expectation value of $\Theta_{\mu\nu}$,

$$\langle\Theta_{\mu\nu}\rangle=(\Phi,\Theta_{\mu\nu}\Phi), \quad (1.30)$$

that satisfies the conservation equation

$$\frac{\partial}{\partial x_\mu}\langle\Theta_{\mu\nu}\rangle=0 \quad (1.31)$$

as a consequence of the identity

$$\begin{aligned}\langle\Theta_{\mu\nu}-T_{\mu\nu}\rangle=\Big\langle&\frac{\partial}{\partial x_\lambda}\tfrac{1}{2}[A_\nu F_{\lambda\mu}+F_{\lambda\mu}A_\nu]\\&-\frac{1}{2}\left[\frac{\partial A_\mu}{\partial x_\lambda}\frac{\partial A_\lambda}{\partial x_\nu}+\frac{\partial A_\lambda}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\lambda}-\delta_{\mu\nu}\frac{\partial A_\lambda}{\partial x_\sigma}\frac{\partial A_\sigma}{\partial x_\lambda}\right]\\&-\frac{i\hbar c}{4}\frac{\partial}{\partial x_\lambda}\tfrac{1}{2}[\bar{\psi}\sigma_{\mu\lambda}\gamma_\nu\psi+\bar{\psi}'\sigma_{\mu\lambda}\gamma_\nu\psi']\Big\rangle, \quad (1.32)\end{aligned}$$

since the supplementary condition is required in the derivation and use of this identity. In Eq. (1.32), $\sigma_{\mu\lambda}$ represents the Dirac matrix spin tensor:

$$\sigma_{\mu\lambda}=(\gamma_\mu\gamma_\lambda-\gamma_\lambda\gamma_\mu)/2i. \quad (1.33)$$

The symmetrical energy-momentum tensor is evidently invariant with respect to gauge transformations and charge conjugation. The simple formula

$$\Theta_{\mu\mu}=-m_0c^2\tfrac{1}{2}(\bar{\psi}\psi+\bar{\psi}'\psi') \quad (1.34)$$

should also be noted.

The spatial volume integrals

$$P_\mu=-\frac{i}{c}\int T_{4\mu}dv \quad (1.35)$$

form a time independent four-vector that unites the momentum and energy integrals of the equations of motion; $P_\mu=(\mathbf{P},iW/c)$. It is a simple consequence of the relation (1.32) that the expectation value of $P_\mu$ can be calculated from either stress tensor. Thus

$$\langle P_\mu\rangle=-\frac{i}{c}\int\langle\Theta_{4\mu}\rangle dv. \quad (1.36)$$

The operators $P_\mu$ form the infinitesimal generators of the coordinate translation group. It is a consequence of the commutation relations that, for example,

$$\frac{i}{\hbar}[A_\mu(x),P_\nu]=\frac{\partial A_\mu(x)}{\partial x_\nu},\qquad \frac{i}{\hbar}[\psi_\alpha(x),P_\nu]=\frac{\partial\psi_\alpha(x)}{\partial x_\nu}. \quad (1.37)$$

More generally, if $F(x)$ is an arbitrary function of the field variables at the point $x$, but does not explicitly involve position coordinates,

$$\frac{i}{\hbar}[F(x),P_\nu]=\frac{\partial F(x)}{\partial x_\nu}. \quad (1.38)$$

One can exploit this aspect of the operators $P_\mu$ to prove anew that they constitute constants of the motion, and to demonstrate that the canonical commutation relations are consistent with the equations of motion. In a similar way, one can introduce other operator constants of the motion which compose the angular momentum tensor. These quantities form the infinitesimal generators of the Lorentz group, and with their aid the covariance of the canonical quantization scheme can be demonstrated. However, it is at this point that we must deviate from the conventional development that here has so briefly been outlined.

The equations of motion and the supplementary condition are manifestly covariant; the canonical commutation relations lack this essential characteristic since a special Lorentz reference system is employed. The commutation relations involve field variables at two points of a four dimensional surface characterized by $t=\text{const.}$ We shall achieve the desired covariance by replacing such surfaces with the invariant concept of a space-like surface. The latter is such that light signals cannot be propagated between any two points on the surface. In terms of the position vectors of two points, $x_\mu$ and $x_\mu'$, it is required that

$$(x_\mu-x_\mu')^2=(\mathbf{r}-\mathbf{r}')^2-c^2(t-t')^2>0, \quad (1.39)$$

which clearly involves no special reference system. Surfaces of the type $t=\text{const.}$ form a special non-covariant class of plane space-like surfaces. The customary commutation relations are essentially an expression of the kinematical independence of field quantities at different points of space for a given time. It is evident that the proper covariant description of this general property should involve field quantities at two space-time points that cannot be connected by light signals, that is, two points on a space-like surface. Accordingly, we endeavor thus to generalize the commutation relations into a manifestly covariant form.

The simplest basis for a generalization of (1.22a) is provided by the two statements that express the properties of $\delta(\mathbf{r}-\mathbf{r}')$:

$$\left[A_\mu(\mathbf{r},t),\frac{1}{c}\frac{\partial}{\partial t}A_\nu(\mathbf{r}',t)\right]=0, \quad \mathbf{r}\neq\mathbf{r}' \quad (1.40\text{a})$$

$$\int\left[A_\mu(\mathbf{r},t),\frac{1}{c}\frac{\partial}{\partial t}A_\nu(\mathbf{r}',t)\right]dv'=i\hbar c\delta_{\mu\nu} \quad (1.40\text{b})$$

in which the spatial volume integration is extended over an arbitrary region that includes the point $\mathbf{r}$. The proper generalization of (1.40a), together with the other vanishing electromagnetic field commutators, is simply

$$[A_\mu(x),A_\nu(x')]=0, \quad (x_\mu-x_\mu')^2>0; \quad (1.41)$$

that is, the field quantities associated with two distinct points on a space-like surface commute. In order to generalize (1.40b), it will prove convenient to define a four-vector differential surface area:

$$d\sigma_\mu=(dx_2dx_3dx_0, dx_1dx_3dx_0, dx_1dx_2dx_0, dx_1dx_2dx_3/i). \quad (1.42)$$

Considered as defining the direction of the normal to a space-like surface, $d\sigma_\mu$ must be a time-like vector, that is, $d\sigma_\mu{}^2<0$. It should be noted that our definitions of surface area and volume are such that the volume generated by the displacement $\delta x_\mu$ imparted to the surface area $d\sigma_\mu$ is $\delta\omega=d\sigma_\mu\delta x_\mu$. It is evident from the notation $dv'=id\sigma_4'$, $\partial A\nu(\mathbf{r}',t)/\partial ct=i\partial A_\nu(x')/\partial x_4'$, that the proper covariant generalization of (1.40b) is

$$\int_\sigma\left[A_\mu(x),\frac{\partial}{\partial x_\lambda'}A_\nu(x')\right]d\sigma_\lambda'=\frac{\hbar c}{i}\delta_{\mu\nu} \quad (1.43)$$

in which the $x'$ integration is extended over an arbitrary portion of a space-like surface $\sigma$ that includes the point $x$.

In order to demonstrate the self-consistency of these and further covariant commutation relations, we must show that the values attributed to such surface integrals are unaltered as the space-like surface $\sigma$ passing through the point $x$ is varied; and, for a fixed surface relative to the point $x$, that the commutation relations are compatible with an arbitrary displacement of $x$. The latter requirement involves a detailed consideration of the equations of motion and will be discussed at an appropriate place. The verification of the first requirement is facilitated by introducing the notion of the functional derivative. The quantity occurring on the left side of (1.43) involves the field variables at all points of the surface $\sigma$ and is thus a functional of the space-like surface $\sigma$, say $F[\sigma]$. We may compare this with the functional of a neighboring space-like surface $\sigma'$, $F[\sigma']$, which surface is such that it deviates from $\sigma$ only in a neighborhood of the point $x$. If the volume enclosed between the surfaces, $\delta\omega$, is allowed to approach zero, we obtain a definition of the functional derivative of $F[\sigma]$ at the point $x$:

$$\frac{\delta F[\sigma]}{\delta\sigma(x)}=\operatorname*{Lim}_{\delta\omega\to 0}\frac{F[\sigma']-F[\sigma]}{\delta\omega}, \tag{1.44}$$

in which the notation emphasises that we are considering the variation in $F$ produced by a deformation of the surface $\sigma$ at the point $x$. A special class of functional, as exemplified by (1.43), is of the form

$$F[\sigma]=\int_\sigma F_\lambda(x')d\sigma_\lambda', \tag{1.45}$$

that is, a surface integral of a point function. The functional derivative has a particularly simple aspect with this type of functional for, according to Gauss' theorem,

$$\frac{\delta F[\sigma]}{\delta\sigma(x)}=\operatorname*{Lim}_{\delta\omega\to 0}\left(\int_{\sigma'}-\int_\sigma\right)F_\lambda(x')d\sigma_\lambda'=\frac{\partial F_\lambda(x)}{\partial x_\lambda}. \tag{1.46}$$

We are now prepared to ascertain the change in (1.43) produced by a deformation of the surface $\sigma$ at the point $x'\neq x$:

$$\frac{\delta}{\delta\sigma(x')}\int_\sigma\left[A_\mu(x),\frac{\partial}{\partial x_\lambda''}A_\nu(x'')\right]d\sigma_\lambda''=[A_\mu(x),\Box'^2A_\nu(x')]=-\frac{1}{c}[A_\mu(x),j_\nu(x')]=0, \tag{1.47}$$

in which the latter statement is a consequence of the covariant expression of the kinematical independence of the quantities associated with the two fields:

$$[A_\mu(x),\psi_\alpha(x')]=[A_\mu(x),\bar\psi_\alpha(x')]=0,\quad (x_\mu-x_\mu')^2>0. \tag{1.48}$$

The corresponding generalizations of the matter field commutation relations are:

$$\{\psi_\alpha(x),\psi_\beta(x')\}=\{\bar\psi_\alpha(x),\bar\psi_\beta(x')\}=\{\psi_\alpha(x),\bar\psi_\beta(x')\}=0,\quad (x_\mu-x_\mu')^2>0, \tag{1.49}$$

and

$$i\int_\sigma\{\psi_\alpha(x),(\bar\psi(x')\gamma_\lambda)_\beta\}d\sigma_\lambda'=\delta_{\alpha\beta}. \tag{1.50}$$

Another version of the last relation, namely

$$i\int_\sigma\{(\gamma_\lambda\psi(x'))_\alpha,\bar\psi_\beta(x)\}d\sigma_\lambda'=\delta_{\alpha\beta} \tag{1.51}$$

can be obtained by taking the Hermitian conjugate of (1.50). To verify that the values given to the surface integrals (1.50) and (1.51) are independent of the particular surface passing through the point $x$, we examine for example,

$$\begin{aligned}\frac{\delta}{\delta\sigma(x')}\int_\sigma\{(\gamma_\lambda\psi(x''))_\alpha,\bar\psi_\beta(x)\}d\sigma_\lambda''&=\left\{\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda'}\psi(x')\right)_\alpha,\bar\psi_\beta(x)\right\}\\&=\frac{ie}{\hbar c}\{(\gamma_\lambda A_\lambda(x')\psi(x'))_\alpha,\bar\psi_\beta(x)\}\\&\quad-\kappa_0\{\psi_\alpha(x'),\bar\psi_\beta(x)\}=0,\end{aligned} \tag{1.52}$$

in view of the vanishing of all such anti-commutators for distinct points of a space-like surface. It can be shown, as before, that the commutation relations are also valid for the charge conjugate matter fields. Thus

$$\begin{aligned}i\int_\sigma\{(\gamma_\lambda\psi'(x'))_\alpha,\bar\psi_\beta'(x)\}d\sigma_\lambda'&=i\int_\sigma\{(\bar\psi(x')\gamma_\lambda C)_\alpha,(C^{-1}\psi(x))_\beta\}d\sigma_\lambda'\\&=C_{\beta\gamma}^{-1}i\int_\sigma\{\psi_\gamma(x),(\bar\psi(x')\gamma_\lambda)_\delta\}d\sigma_\lambda'C_{\delta\alpha}\\&=(C^{-1}C)_{\beta\alpha}=\delta_{\alpha\beta},\end{aligned} \tag{1.53}$$

in virtue of (1.50). In the course of these rearrangements, the following relation has also been employed

$$(\gamma_\lambda C)^T=\gamma_\lambda C, \tag{1.54}$$

which is an immediate consequence of the properties of the matrix $C$.

An obviously covariant definition of the energy-momentum four-vector, replacing Eq. (1.35), is

$$P_\mu=\frac{1}{c}\int_\sigma d\sigma_\lambda T_{\lambda\mu}(x), \tag{1.55}$$

in which the integration is extended over an entire space-like surface. The conservation laws now have their covariant expression in the statement that $P_\mu$ is independent of the surface $\sigma$. Thus

$$\frac{\delta}{\delta\sigma(x)}P_\mu c=\frac{\partial}{\partial x_\lambda}T_{\lambda\mu}(x)=0. \tag{1.56}$$

The conservation law for the total charge

$$Q=\frac{1}{c}\int_\sigma d\sigma_\mu j_\mu(x)$$

has an analogous expression:

$$\frac{\delta}{\delta\sigma(x)}Qc=\frac{\partial}{\partial x_\mu}j_\mu(x)=0. \tag{1.57}$$

It may be instructive to verify that the commutation relations expressing the displacement operator interpretation of the $P_\mu$ emerge from the covariant commutation rules that have been developed. For this purpose, the following lemma is useful:

$$\int_\sigma\left[d\sigma_\mu\frac{\partial}{\partial x_\nu}F(x)-d\sigma_\nu\frac{\partial}{\partial x_\mu}F(x)\right]=0. \tag{1.58}$$

The proof of the lemma is given by remarking that the surface integral is independent of $\sigma$:

$$\frac{\delta}{\delta\sigma(x)}\int_\sigma\left[d\sigma_\mu'\frac{\partial}{\partial x_\nu'}F(x')-d\sigma_\nu'\frac{\partial}{\partial x_\mu'}F(x')\right]$$
$$=\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}F(x)-\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}F(x)=0, \tag{1.59}$$

and that, for the particular surface $t=\text{const.}$, the only components of (1.58) that are not identically zero are $\mu=4$, $\nu=k=1, 2, 3$, say. But

$$i\int d\sigma_4\frac{\partial}{\partial x_k}F(x)=\int dv\frac{\partial}{\partial x_k}F(x)=0 \tag{1.60}$$

for a closed system.

In order to express the operator $P_\nu$ in terms of $\psi$ and $\bar\psi$ alone, we note that, according to (1.12),

$$\bar\psi'\gamma_\mu\frac{\partial\psi'}{\partial x_\nu}=\psi\gamma_\mu{}^T\frac{\partial\bar\psi}{\partial x_\nu}=\psi\frac{\partial\bar\psi}{\partial x_\nu}\gamma_\mu, \tag{1.61}$$

whence

$$\int_\sigma d\sigma_\mu\bar\psi'\gamma_\mu\frac{\partial\psi'}{\partial x_\nu}=-\int_\sigma d\sigma_\mu\frac{\partial\psi}{\partial x_\nu}\bar\psi\gamma_\mu$$
$$+\int_\sigma d\sigma_\mu\frac{\partial}{\partial x_\nu}(\psi\bar\psi\gamma_\mu). \tag{1.62}$$

The lemma (1.58) now assures us that

$$\int_\sigma d\sigma_\mu\frac{\partial}{\partial x_\nu}(\psi\bar\psi\gamma_\mu)=\int_\sigma d\sigma_\nu\frac{\partial}{\partial x_\mu}(\psi\bar\psi\gamma_\mu)=0, \tag{1.63}$$

in which the latter statement involves the charge conservation equation. Therefore, $P_\nu$ can be written

$$P_\nu=\frac{1}{2c}\int_\sigma d\sigma_\mu\left[\frac{\partial A_\lambda}{\partial x_\mu}\frac{\partial A_\lambda}{\partial x_\nu}+\frac{\partial A_\lambda}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\mu}-\delta_{\mu\nu}\left(\frac{\partial A_\lambda}{\partial x_\sigma}\right)^2\right]$$
$$+\frac{\hbar}{2}\int_\sigma d\sigma_\mu\left[\bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\psi}{\partial x_\nu}\bar\psi\gamma_\mu\right], \tag{1.64}$$

whence

$$\frac{i}{\hbar}[\psi_\alpha(x),P_\nu]=i\int_\sigma d\sigma_\mu'\{\psi_\alpha(x),(\bar\psi(x')\gamma_\mu)_\beta\}$$
$$\times\frac{\partial\psi_\beta(x')}{\partial x_\nu'}=\frac{\partial\psi_\alpha(x)}{\partial x_\nu}, \tag{1.65}$$

and

$$\frac{i}{\hbar}[A_\mu(x),P_\nu]=\frac{i}{\hbar c}\int_\sigma d\sigma_\lambda'\left[A_\mu(x),\frac{\partial A_\sigma(x')}{\partial x_\lambda'}\right]$$
$$\times\frac{\partial A_\sigma(x')}{\partial x_\nu'}+\frac{i}{\hbar c}\int_\sigma\left(d\sigma_\lambda'\left[A_\mu(x),\frac{\partial A_\sigma(x')}{\partial x_\nu'}\right]\right.$$
$$\left.-d\sigma_\nu'\left[A_\mu(x),\frac{\partial A_\sigma(x')}{\partial x_\lambda'}\right]\right)\frac{\partial A_\sigma(x')}{\partial x_\lambda'}$$
$$=\frac{\partial A_\mu(x)}{\partial x_\nu}. \tag{1.66}$$

In the latter proof, the lemma (1.58) has been used, in addition to the properties of commutators.

It can now be shown that the covariant commutation relations are consistent with the equations of motion. We examine the change in the commutator or anticommutator of two field variables associated with two points on a space-like surface, produced by a rigid displacement of the surface. In other words, we seek to evaluate

$$\frac{\partial}{\partial x_\nu}[F(x),G(x-\xi)] \quad \text{or} \quad \frac{\partial}{\partial x_\nu}\{F(x),G(x-\xi)\},$$

where $\xi_\mu$ is a space-like vector and $F$, $G$ are any two field variables. It is a consequence of elementary identities that if $F$ and $G$ obey the equations of motion (1.38), so also do the brackets

$$[F(x),G(x-\xi)] \quad \text{and} \quad \{F(x),G(x-\xi)\}.$$

Therefore, the specification of such brackets as $\xi$-dependent multiples of the unit operator is self-consistent, since both the derivative with respect to $x_\nu$, and the commutator with $P_\nu$, vanish.

The formulation of quantum mechanics that has now been developed is obviously covariant in all its aspects. However, it is not entirely suitable as a practical means of treating electrodynamic questions. In the course of application, it is often necessary to evaluate commutators of field quantities at points separated by a time-like interval. Such commutators are to be constructed by solving the equations of motion subject to boundary conditions on a space-like surface. This jumbling of the kinematical and dynamical aspects of the situation is a detriment in the systematic discussion of electrodynamic problems. At the opposite extreme is the Schrödinger picture, in which all operators are time independent, and the time development of the system is represented by a varying state vector; a procedure that is non-covariant in its aspect. We now seek a formulation that enables us to retain the evident covariance of the Heisenberg representation, and yet offers something akin to the advantage of the Schrödinger representation, a distinct separation between kinematical and dynamical aspects. The desired separation is to be found in that between the elementary properties of non-interacting fields, and the modification of these properties by the coupling between fields. For non-interacting fields, it is a simple matter to carry out the program previously mentioned, and construct commutation relations for field quantities at arbitrary space-time points. In order to exploit this advantage, it is necessary to find a canonical transformation that changes the equations of motion for field quantities in the Heisenberg representation into those of non-interacting fields, and therefore describes the coupling between fields in terms of a varying state vector. We shall perform this transformation in the next section, and thus obtain an obviously covariant and practical form of quantum electrodynamics, expressed in a mixed Heisenberg-Schrödinger representation, which may be called the interaction representation.[11]

## 2. THE INTERACTION REPRESENTATION

To alter the equations of motion in the above outlined manner, we introduce a unitary operator $U[\sigma]$, defined for a space-like surface $\sigma$, and construct the state vector of the interaction representation

$$\Psi[\sigma]=U[\sigma]\Phi, \tag{2.1}$$

which depends upon the surface $\sigma$, in contrast with the constant vector $\Phi$ of the Heisenberg representation. The expectation value of some field variable $\mathbf{F}(x)$ becomes (in this section, the operators of the Heisenberg representation will be denoted by bold face letters)

$$\begin{aligned}(\Phi,\mathbf{F}(x)\Phi)&=(\Psi[\sigma],U[\sigma]\mathbf{F}(x)U^{-1}[\sigma]\Psi[\sigma])\\&=(\Psi[\sigma],F(x)\Psi[\sigma]),\end{aligned} \tag{2.2}$$

which defines the operators of the interaction representation in terms of those of the Heisenberg representation:

$$F(x)=U[\sigma]\mathbf{F}(x)U^{-1}[\sigma]. \tag{2.3}$$

It is understood that $\sigma$ is a space-like surface passing through the point $x$. In order, however, that $F(x)$ depend only on the point $x$ and not on the particular surface $\sigma$, the form of $U[\sigma]$ must be restricted, as indicated by the following

[11] The interaction representation can be regarded as a field generalization of the many-time formalism, from which point of view it has already been considered by S. Tomonaga, Prog. Theor. Phys. **1**, 27 (1946). Relativistic quantum theories have also been discussed recently by P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948).

requirement:

$$\frac{\delta}{\delta\sigma(x')}F(x)=\frac{\delta U[\sigma]}{\delta\sigma(x')}\mathbf{F}(x)U^{-1}[\sigma]$$
$$-U[\sigma]\mathbf{F}(x)U^{-1}[\sigma]\frac{\delta U[\sigma]}{\delta\sigma(x')}U^{-1}[\sigma]$$
$$=\left[\frac{\delta U[\sigma]}{\delta\sigma(x')}U^{-1}[\sigma],F(x)\right]=0,$$
$$(x_\mu-x_\mu')^2>0. \quad (2.4)$$

This will be satisfied if

$$\frac{\delta U[\sigma]}{\delta\sigma(x')}U^{-1}[\sigma]$$

is an invariant function of the field operators at the point $x'$, since the commutation properties on the surface $\sigma$ are unaffected by the unitary transformation. If, further, the unitary character of $U[\sigma]$ is to be preserved by its equation of motion, it is necessary that

$$i\frac{\delta U[\sigma]}{\delta\sigma(x)}U^{-1}[\sigma]$$

be a Hermitian operator. Therefore, on writing

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(x)}=\mathcal{H}(x)U[\sigma], \quad (2.5)$$

we obtain a covariant equation of motion for $U[\sigma]$, in which $\mathcal{H}(x)$ is a Hermitian operator, an invariant function of the field quantities at the point $x$, and has the dimensions of an energy density. The equation of motion for $\Psi[\sigma]$ is, correspondingly,

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\mathcal{H}(x)\Psi[\sigma]. \quad (2.6)$$

We have obtained the conditions that must be satisfied by any canonical transformation. It will now be shown that the special transformation desired is attained with $\mathcal{H}(x)$ chosen as the negative of the coupling term in the Lagrangian density, that is,

$$\mathcal{H}(x)=-(1/c)j_\mu(x)A_\mu(x). \quad (2.7)$$

To construct the equations of motion in the interaction representation, we first note that the gradient of any field quantity can be exhibited as a functional derivative, through an obvious generalization of Gauss' theorem:

$$\frac{\partial F(x)}{\partial x_\nu}=\frac{\delta}{\delta\sigma(x)}\int_\sigma F(x')d\sigma_\nu'$$
$$=\frac{\delta}{\delta\sigma(x)}U[\sigma]\int_\sigma \mathbf{F}(x')d\sigma_\nu'U^{-1}[\sigma]. \quad (2.8)$$

The functional derivative in the latter form affects both the surface of integration and the operator $U[\sigma]$, whence

$$\frac{\partial F(x)}{\partial x_\nu}=U[\sigma]\frac{\partial \mathbf{F}(x)}{\partial x_\nu}U^{-1}[\sigma]$$
$$+\int_\sigma\left[\frac{\delta U[\sigma]}{\delta\sigma(x)}U^{-1}[\sigma],F(x')\right]d\sigma_\nu'$$
$$=U[\sigma]\frac{\partial \mathbf{F}(x)}{\partial x_\nu}U^{-1}[\sigma]$$
$$-\frac{i}{\hbar c}\int_\sigma[\mathcal{H}(x),F(x')]d\sigma_\nu'. \quad (2.9)$$

If we first place $F(x)=A_\mu(x)$, it is immediately found from the covariant commutation relations on the space-like surface $\sigma$, that

$$\frac{\partial A_\mu(x)}{\partial x_\nu}=U[\sigma]\frac{\partial \mathbf{A}_\mu(x)}{\partial x_\nu}U^{-1}[\sigma], \quad (2.10)$$

which, indeed, is necessary, in order that the electromagnetic field commutation relations retain their form under this canonical transformation. However, with $F(x)=\partial A_\mu(x)/\partial x_\nu$, one obtains

$$\square^2 A_\mu(x)=U[\sigma]\square^2\mathbf{A}_\mu(x)U^{-1}[\sigma]$$
$$+\frac{1}{c}j_\lambda(x)\frac{i}{\hbar c}\int_\sigma\left[A_\lambda(x),\frac{\partial A_\mu(x')}{\partial x_\nu'}\right]d\sigma_\nu'$$
$$=-\frac{1}{c}U[\sigma]\mathbf{j}_\mu(x)U^{-1}[\sigma]+\frac{1}{c}j_\mu(x)$$
$$=0; \quad (2.11)$$

the equations of motion for the electromagnetic field in the interaction representation are those of an isolated field. In addition, the supplementary condition is unchanged in form:

$$\frac{\partial A_\mu(x)}{\partial x_\mu}\Psi[\sigma]=0 \tag{2.12}$$

provided the point $x$ lies on the surface $\sigma$. Finally, if $F(x)=\gamma_\nu\psi(x)$,

$$\begin{aligned}\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+\kappa_0\right)\psi(x)&\\ &=U[\sigma]\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+\kappa_0\right)\psi(x)\,U^{-1}[\sigma]\\ &\quad+\frac{1}{c}A_\mu(x)\frac{i}{\hbar c}\int_\sigma[j_\mu(x),\gamma_\nu\psi(x')]d\sigma_\nu'.\end{aligned} \tag{2.13}$$

But, according to (1.12) and (1.14)

$${}_\mu(x)=\frac{iec}{2}[\bar{\psi}(x)\gamma_\mu\psi(x)-\psi(x)\bar{\psi}(x)\gamma_\mu], \tag{2.14}$$

and

$$\begin{aligned}&[j_\mu(x),\gamma_\nu\psi(x')]\\ &\qquad=-iec\{\gamma_\nu\psi(x'),\bar{\psi}_\alpha(x)\}(\gamma_\mu\psi(x))_\alpha,\end{aligned} \tag{2.15}$$

so that

$$\begin{aligned}&\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+\kappa_0\right)\psi(x)\\ &\quad=\frac{ie}{\hbar c}\gamma_\mu A_\mu(x)\psi(x)-i\int_\sigma\{\gamma_\nu\psi(x'),\bar{\psi}_\alpha(x)\}\\ &\qquad\qquad\times d\sigma_\nu'\left(\frac{ie}{\hbar c}\gamma_\mu A_\mu(x)\psi(x)\right)_\alpha\\ &\quad=0;\end{aligned} \tag{2.16}$$

the equations of motion for the matter field in the interaction representation are those of an isolated field. This completely proves the correctness of the choice (2.7) for $\mathcal{H}(x)$.

We may now proceed to construct the general commutation laws for the field quantities in the new representation, by employing their elementary equations of motion. This process will be facilitated by introducing two invariant functions of position, $D(x)$ and $\Delta(x)$, which are associated with the electromagnetic and matter fields, respectively, and have the following covariant definitions:

$$\begin{gathered}\Box^2 D(x)=0;\quad D(x)=0,\quad x_\mu^2>0\\ \int_\sigma\frac{\partial D(x)}{\partial x_\mu}d\sigma_\mu=1,\end{gathered} \tag{2.17}$$

$$\begin{gathered}(\Box^2-\kappa_0^2)\Delta(x)=0;\quad \Delta(x)=0,\quad x_\mu^2>0\\ \int_\sigma\frac{\partial\Delta(x)}{\partial x_\mu}d\sigma_\mu=1.\end{gathered} \tag{2.18}$$

In these definitions, the surface integrations are to be extended over a space-like surface that includes the origin. It is easily verified that the constant value attributed to the surface integrals for arbitrary $\sigma$ is consistent with the other equations. The detailed construction of these and related functions will be postponed to the second paper of this series; the properties contained in the equations of definition will suffice for our present purposes. It is easily deduced for example, that $D$ and $\Delta$ are odd functions of the coordinates:

$$D(-x)=-D(x),\quad \Delta(-x)=-\Delta(x). \tag{2.19}$$

We note that

$$\begin{aligned}&\frac{\delta}{\delta\sigma(x)}\int_\sigma\left[\Delta(x-x')\frac{\partial}{\partial x_\mu}\Delta(x-x'')\right.\\ &\qquad\qquad\left.-\Delta(x-x'')\frac{\partial}{\partial x_\mu}\Delta(x-x')\right]d\sigma_\mu\\ &\quad=\Delta(x-x')(\Box^2-\kappa_0^2)\Delta(x-x'')\\ &\qquad\qquad-\Delta(x-x'')(\Box^2-\kappa_0^2)\Delta(x-x')\\ &\quad=0,\end{aligned} \tag{2.20}$$

which implies that the surface integral is independent of the particular surface $\sigma$. By choosing $\sigma$ to be, successively, a space-like surface through the points $x'$ and $x''$, it is inferred that

$$\Delta(x''-x')=-\Delta(x'-x'') \tag{2.21}$$

which proves the second statement of (2.19). The proof for $D(x)$ is identical.

The importance of these invariant functions stems from their utility in expressing the solutions of the equations of motion in terms of boundary values prescribed on some space-like surface. The electromagnetic potentials are uniquely determined if $A_\mu(x)$ and its normal derivative are specified on a surface $\sigma$. The explicit realization of this relation is provided by

$$A_\mu(x)=\int_\sigma\bigg[D(x-x')\frac{\partial}{\partial x_\nu'}A_\mu(x')-A_\mu(x')\frac{\partial}{\partial x_\nu'}D(x-x')\bigg]d\sigma_\nu'. \quad (2.22)$$

To verify this statement, it is sufficient to observe that, analogously to Eq. (2.20), the right side of (2.22) is independent of $\sigma$, which can be specially chosen as a space-like surface through the point $x$, yielding

$$\int_\sigma A_\mu(x')\frac{\partial}{\partial x_\nu'}D(x'-x)d\sigma_\nu'=A_\mu(x)$$

as the value of the surface integral. The corresponding solution of the boundary value problem for the first order Dirac equation is provided by

$$\psi(x)=\int_\sigma S(x-x')\gamma_\mu\psi(x')d\sigma_\mu', \quad (2.23)$$

where

$$S(x)=\bigg(\gamma_\nu\frac{\partial}{\partial x_\nu}-\kappa_0\bigg)\Delta(x). \quad (2.24)$$

Following the general pattern, we remark that the right side of (2.23) is independent of $\sigma$:

$$\begin{aligned}\frac{\delta}{\delta\sigma(x')}\int_\sigma S(x-x')\gamma_\mu\psi(x')d\sigma_\mu' &=\frac{\partial}{\partial x_\mu'}(S(x-x')\gamma_\mu\psi(x'))\\ &=S(x-x')\bigg(\gamma_\mu\frac{\partial}{\partial x_\mu'}+\kappa_0\bigg)\psi(x')-\bigg(\frac{\partial}{\partial x_\mu}S(x-x')\gamma_\mu+\kappa_0S(x-x')\bigg)\psi(x')\\ &=0, \end{aligned}\quad (2.25)$$

since

$$\bigg(\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa_0\bigg)S(x)=\frac{\partial}{\partial x_\mu}S(x)\gamma_\mu+\kappa_0S(x)=(\Box^2-\kappa_0^2)\Delta(x)=0; \quad (2.26)$$

and that an evaluation with a surface through the point $x$, gives

$$\begin{aligned}\int_\sigma\gamma_\nu\gamma_\mu\frac{\partial}{\partial x_\nu}\Delta(x-x')\psi(x')d\sigma_\mu' &=\int_\sigma\frac{\partial}{\partial x_\mu'}\Delta(x'-x)\psi(x')d\sigma_\mu'\\ &+\frac{i}{2}\sigma_{\nu\mu}\int_\sigma\bigg[\frac{\partial}{\partial x_\nu'}\Delta(x'-x)d\sigma_\mu'-\frac{\partial}{\partial x_\mu'}\Delta(x'-x)d\sigma_\nu'\bigg]\psi(x')=\psi(x)\end{aligned}$$

with the aid of the lemma (1.58). The adjoint equation

$$\bar\psi(x)=\int_\sigma d\sigma_\mu'\bar\psi(x')\gamma_\mu S(x'-x) \quad (2.27)$$

can be proved directly, or inferred from (2.23).

The construction of the general commutation relations is now trivial. To evaluate $[A_\mu(x), A_\nu(x')]$, for example, it is merely necessary to express $A_\mu(x)$ in terms of the field variables on a space-like surface that includes the point $x'$, and employ the commutation relations for such surfaces. Thus

$$[A_\mu(x),A_\nu(x')]=-\int_\sigma D(x-x'')\times\bigg[A_\nu(x'),\frac{\partial}{\partial x_\lambda''}A_\mu(x'')\bigg]d\sigma_\lambda'',$$

whence

$$[A_\mu(x),A_\nu(x')]=i\hbar c\delta_{\mu\nu}D(x-x'). \quad (2.28)$$

In a similar way,

$$\{\psi_\alpha(x),\bar\psi_\beta(x')\}=\int_\sigma S_{\alpha\gamma}(x-x'')\times\{(\gamma_\mu\psi(x''))_\gamma,\bar\psi_\beta(x')\}d\sigma_\mu''$$

so that

$$\{\psi_\alpha(x),\bar\psi_\beta(x')\}=\frac{1}{i}S_{\alpha\beta}(x-x')=\frac{1}{i}\bigg(\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa_0\bigg)_{\alpha\beta}\Delta(x-x'). \quad (2.29)$$

All other matter field anti-commutators vanish. Of course, the matter field commutation relations are invariant with respect to charge conjugation.

Finally, we turn to the generalization of the supplementary condition (2.12), which consists of removing the restriction that $x$ be situated on the surface $\sigma$. It follows from (2.22) that, for an arbitrary point $x$,

$$\frac{\partial A_\mu(x)}{\partial x_\mu}\Psi[\sigma]=\int_\sigma D(x-x'')\times\frac{\partial}{\partial x_\nu''}\left(\frac{\partial A_\mu(x'')}{\partial x_\mu''}\right)d\sigma_\nu''\Psi[\sigma]. \quad (2.30)$$

However, according to (2.9), with $F=\partial A_\mu(x'')/\partial x_\mu''$,

$$\begin{aligned}\frac{\partial}{\partial x_\nu''}\left(\frac{\partial A_\mu(x'')}{\partial x_\mu''}\right)\Psi[\sigma]&=U[\sigma]\frac{\partial}{\partial x_\nu''}\left(\frac{\partial \mathbf{A}_\mu(x'')}{\partial x_\mu''}\right)\Phi-\frac{i}{\hbar c}\int_\sigma\left[\mathcal{H}(x''),\frac{\partial A_\mu(x')}{\partial x_\mu'}\right]d\sigma_\nu'\Psi[\sigma]\\&=\frac{i}{\hbar c}\int_\sigma\left[A_\lambda(x''),\frac{\partial A_\mu(x')}{\partial x_\mu'}\right]\times d\sigma_\nu'\frac{1}{c}j_\lambda(x'')\Psi[\sigma]\\&=\frac{i}{\hbar c}\int_\sigma\left[A_\mu(x'),\frac{\partial A_\lambda(x'')}{\partial x_\nu''}\right]\times d\sigma_\mu'\frac{1}{c}j_\lambda(x'')\Psi[\sigma].\end{aligned} \quad (2.31)$$

In the last transformation, we have used the lemma (1.58) and the fact that the electromagnetic field commutators contain only the difference in coordinates of the two points involved. On introducing (2.31) into (2.30) and performing the $x''$ integration, we find without further difficulty that

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}-\int_\sigma D(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu'\right]\Psi[\sigma]=0, \quad (2.32)$$

which is the supplementary condition for the interaction representation. Although the consistency of the supplementary condition is guaranteed by the corresponding property in the Heisenberg representation, it is well to verify it directly. However, since the proof involves the commutation properties of the current four-vector, we digress briefly to derive the necessary theorems.

It is easy to deduce from the expression (2.14) for $j_\mu(x)$, and the anti-commutator (2.29), that

$$\begin{aligned}[j_\mu(x),j_\nu(x')]&=ie^2c^2[\bar\psi(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x')\\&\quad-\bar\psi(x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x)].\end{aligned} \quad (2.33)$$

Of course, all components of $j_\mu$ commute at two distinct points on a space-like surface. However, the important statement is that a time-like component of $j_\mu$ commutes with all components of the current at the same point. We prove this by demonstrating that

$$\int_\sigma[j_\mu(x),j_\nu(x')]d\sigma_\nu'=\frac{1}{c}[j_\mu(x),Q]=0 \quad (2.34)$$

where $\sigma$ is any space-like surface, which, in particular, can include the point $x$. The validity of this statement follows immediately from (2.23) and (2.27), since

$$\begin{aligned}\int_\sigma[j_\mu(x),j_\nu(x')]d\sigma_\nu'&=ie^2c^2[\bar\psi(x)\gamma_\mu\psi(x)-\bar\psi(x)\gamma_\mu\psi(x)]\\&=0.\end{aligned} \quad (2.35)$$

Indeed, Eq. (2.34) is an expression of charge conservation, for, according to (2,9), with $F=j_\nu(x)$:

$$\begin{aligned}\frac{\partial j_\nu(x)}{\partial x_\nu}&=\frac{i}{\hbar c}\frac{1}{c}\int_\sigma[j_\mu(x),j_\nu(x')]d\sigma_\nu'A_\mu(x)\\&=\frac{i}{\hbar c}[j_\mu(x),Q]A_\mu(x).\end{aligned} \quad (2.36)$$

To prove the suitability of (2.32) as a supplementary condition, we must show that it is consistent with the field equations of motion, the equation of motion for $\Psi[\sigma]$, and the commutation relations. In terms of the operator

$$\Omega[x,\sigma]=\frac{\partial A_\mu(x)}{\partial x_\mu}-\int_\sigma D(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu', \quad (2.37)$$

we must verify that

$$\square^2\Omega[x,\sigma]=0, \tag{2.38a}$$

$$i\hbar c\frac{\delta\Omega[x,\sigma]}{\delta\sigma(x')}+[\Omega[x,\sigma],\mathcal{H}(x')]=0, \tag{2.38b}$$

$$[\Omega[x,\sigma],\Omega[x',\sigma]]=0. \tag{2.38c}$$

The first statement is trivial. As to (2.38b), note that

$$i\hbar c\frac{\delta\Omega[x,\sigma]}{\delta\sigma(x')}=i\hbar\frac{\partial D(x-x')}{\partial x_\mu}j_\mu(x')$$

while

$$[\Omega[x,\sigma],\mathcal{H}(x')]=-\frac{1}{c}\left[\frac{\partial A_\mu(x)}{\partial x_\mu},A_\nu(x')\right]j_\nu(x')$$

$$=-i\hbar\frac{\partial D(x-x')}{\partial x_\mu}j_\mu(x')$$

in view of the property of $j_\mu(x)$ just established. Finally, the same property implies that

$$[\Omega[x,\sigma],\Omega[x',\sigma]]=\left[\frac{\partial A_\mu(x)}{\partial x_\mu},\frac{\partial A_\nu(x')}{\partial x_\nu'}\right]$$

$$=-i\hbar c\square^2D(x-x')=0.$$

Gauge invariance has a different aspect in the new representation from that of the Heisenberg representation, since the matter field equations do not involve the electromagnetic field. On introducing a change in gauge

$$A_\mu(x)\rightarrow A_\mu(x)-\frac{\partial\Lambda(x)}{\partial x_\mu},$$

where $\Lambda(x)$ is a scalar function of position such that

$$\square^2\Lambda(x)=0,$$

the supplementary condition, commutation relations and field equations of motion are unaffected, but the equation of motion for $\Psi[\sigma]$ becomes

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}$$

$$=\left\{\mathcal{H}(x)+\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}j_\mu(x)\Lambda(x)\right)\right\}\Psi[\sigma], \tag{2.39}$$

in which the charge conservation equation has been used. We shall show that it is possible to restore this equation to its original form, and thus prove gauge invariance, by a canonical transformation on $\Psi[\sigma]$. Indeed, the proper transformation is

$$\Psi[\sigma]\rightarrow e^{-iG[\sigma]}\Psi[\sigma] \tag{2.40}$$

where

$$G[\sigma]=\frac{1}{\hbar c}\int_\sigma\frac{1}{c}j_\mu(x)\Lambda(x)d\sigma_\mu. \tag{2.41}$$

The equation of motion for the new state vector is

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}+i\hbar ce^{iG[\sigma]}\frac{\delta e^{-iG[\sigma]}}{\delta\sigma(x)}\Psi[\sigma]$$

$$=\left\{\mathcal{H}(x)+\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}j_\mu(x)\Lambda(x)\right)\right\}\Psi[\sigma], \tag{2.42}$$

as a consequence of the commutation properties of $j_\mu$ on a space-like surface. We may now employ the simple expansion theorem

$$e^{iG}\frac{\delta e^{-iG}}{\delta\sigma(x)}=-i\frac{\delta G}{\delta\sigma(x)}+\frac{1}{2!}\left[G,\frac{\delta G}{\delta\sigma(x)}\right]$$

$$+\frac{i}{3!}\left[G,\left[G,\frac{\delta G}{\delta\sigma(x)}\right]\right]+\cdots \tag{2.43}$$

to deduce that

$$i\hbar ce^{iG}\frac{\delta e^{-iG}}{\delta\sigma(x)}=\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}j_\mu(x)\Lambda(x)\right)$$

$$+\frac{i}{2c}[G,j_\mu(x)]\frac{\partial\Lambda(x)}{\partial x_\mu}+\cdots=\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}j_\mu(x)\Lambda(x)\right),$$

in which the commutability of $j_\mu$ with a time-like component of $j_\mu$ on the surface $\sigma$ ensures that only the first term of the series survives. We have thereby demonstrated the correctness of the transformation (2.40).

The form of the energy-momentum quantities, as well as their significance as displacement operators, is altered by the canonical transformation that generates the interaction representation. In the Heisenberg representation, the functional derivative of an operator is of immediate significance in computing the functional derivative of the expectation value of that operator:

$$\frac{\delta}{\delta\sigma(x)}(\Phi,\mathbf{F}[\sigma]\Phi)=\left(\Phi,\frac{\delta\mathbf{F}[\sigma]}{\delta\sigma(x)}\Phi\right). \quad (2.44)$$

In the interaction representation, however, part of the change in the expectation value is accounted for by the variation in $\Psi[\sigma]$:

$$\frac{\delta}{\delta\sigma(x)}(\Psi[\sigma],F[\sigma]\Psi[\sigma])=\left(\Psi[\sigma],\frac{\delta F[\sigma]}{\delta\sigma(x)}\Psi[\sigma]\right)+\frac{i}{\hbar c}(\Psi[\sigma],[\mathcal{H}(x),F[\sigma]]\Psi[\sigma]). \quad (2.45)$$

Accordingly, it is natural to define the total functional derivative of an operator,

$$\frac{\Delta F[\sigma]}{\Delta\sigma(x)}=\frac{\delta F[\sigma]}{\delta\sigma(x)}+\frac{i}{\hbar c}[\mathcal{H}(x),F[\sigma]]=U[\sigma]\frac{\delta\mathbf{F}[\sigma]}{\delta\sigma(x)}U^{-1}[\sigma] \quad (2.46)$$

which is composed of the partial functional derivatives, expressing the explicit coordinate variation and the implicit dynamical variation. With this definition,

$$\frac{\delta}{\delta\sigma(x)}(\Psi[\sigma],F[\sigma]\Psi[\sigma])=\left(\Psi[\sigma],\frac{\Delta F[\sigma]}{\Delta\sigma(x)}\Psi[\sigma]\right). \quad (2.47)$$

If the functional is of the form

$$F_\mu[\sigma]=\int_\sigma F(x)d\sigma_\mu,$$

we are led to write

$$\frac{\Delta F_\mu[\sigma]}{\Delta\sigma(x)}=\frac{dF(x)}{dx_\mu},$$

where

$$\frac{dF(x)}{dx_\mu}=\frac{\partial F(x)}{\partial x_\mu}+\frac{i}{\hbar c}\left[\mathcal{H}(x),\int_\sigma F(x')d\sigma_\mu'\right]$$

defines the total coordinate derivative. It should be clear that the conservation theorem (1.56) and the equation of motion (1.38), in the interaction representation are to be written

$$\frac{\Delta P_\mu[\sigma]}{\Delta\sigma(x)}=\frac{\delta P_\mu[\sigma]}{\delta\sigma(x)}+\frac{i}{\hbar c}[\mathcal{H}(x),P_\mu[\sigma]]=0, \quad (2.48)$$

and

$$\frac{i}{\hbar}[F(x),P_\mu[\sigma]]=\frac{dF(x)}{dx_\mu}=\frac{\partial F(x)}{\partial x_\mu}+\frac{i}{\hbar c}\left[\mathcal{H}(x),\int_\sigma F(x')d\sigma_\mu'\right]. \quad (2.49)$$

Now the partial coordinate derivative $\partial F(x)/\partial x_\mu$ is that to be associated with the behavior of non-interacting fields, and can therefore be calculated from the energy-momentum four-vector of the isolated fields, $P_\mu^{(0)}$, according to

$$\frac{i}{\hbar}[F(x),P_\mu^{(0)}]=\frac{\partial F(x)}{\partial x_\mu}. \quad (2.50)$$

Therefore,

$$[F(x),P_\mu[\sigma]-P_\mu^{(0)}]=\frac{1}{c}\int_\sigma[\mathcal{H}(x),F(x')]d\sigma_\mu'=-\frac{1}{c}\int_\sigma[F(x),\mathcal{H}(x')]d\sigma_\mu' \quad (2.51)$$

in which we have used the fact that only the point $x'=x$ will contribute to the surface integral. One may infer that

$$P_\mu[\sigma]=P_\mu^{(0)}-\frac{1}{c}\int_\sigma\mathcal{H}(x)d\sigma_\mu, \quad (2.52)$$

which, indeed, is compatible with the conservation theorem (2.48), since

$$\frac{\Delta P_\mu[\sigma]}{\Delta\sigma(x)}=\frac{\delta P_\mu^{(0)}}{\delta\sigma(x)}-\frac{1}{c}\left(\frac{\partial\mathcal{H}(x)}{\partial x_\mu}-\frac{i}{\hbar}[\mathcal{H}(x),P_\mu^{(0)}]\right)=0. \quad (2.53)$$

The statement (2.52) can be confirmed by direct calculation. The appropriate transcription of the Heisenberg operator (1.64) involves the introduction of the total derivatives $dA_\lambda/dx_\nu$ and $d\psi/dx_\nu$. Only the latter differs from the explicit coordinate derivative. Now the operator $P_\nu^{(0)}$ is formally identical with (1.64), but expressed in terms of the interaction representation operators and their explicit coordinate derivatives. Therefore,

$$P_\nu[\sigma]=P_\nu^{(0)}+\frac{i}{2c}\int_\sigma d\sigma_\mu \times[\bar{\psi}(x),[\mathcal{H}(x),\gamma_\mu\psi(x')]]d\sigma_\nu'. \quad (2.54)$$

However,

$$[\mathcal{H}(x),\gamma_\mu\psi(x')]=-\frac{1}{c}[j_\lambda(x),\gamma_\mu\psi(x')]A_\lambda(x)$$
$$=ie\{\gamma_\mu\psi(x'),\bar{\psi}_\alpha(x)\}(\gamma_\lambda\psi(x))_\alpha A_\lambda(x), \quad (2.55)$$

whence

$$P_\nu[\sigma]-P_\nu^{(0)}$$
$$=\frac{ie}{2c}\int_\sigma d\sigma_\mu i\{\gamma_\mu\psi(x'),\bar{\psi}_\alpha(x)\} [\bar{\psi}(x),(\gamma_\lambda\psi(x))_\alpha]A_\lambda(x)d\sigma_\nu'$$
$$=\frac{ie}{2c}\int_\sigma[\bar{\psi}(x)\gamma_\lambda\psi(x)-\gamma_\lambda\psi(x)\bar{\psi}(x)]A_\lambda(x)d\sigma_\nu$$
$$=\frac{1}{c^2}\int_\sigma j_\lambda(x)A_\lambda(x)d\sigma_\nu, \quad (2.56)$$

which is the content of Eq. (2.52).

## 3. COVARIANT ELIMINATION OF THE LONGITUDINAL FIELD

It is the function of the supplementary condition to ensure that the electromagnetic field contains no spin-less light quanta, which have various unphysical properties. It is possible, indeed to eliminate the scalar potential and the longitudinal part of the vector potential, leaving only the transverse vector potential as the quantity truly descriptive of light waves. Such conventional procedures suffer from a lack of covariance which will be remedied in this section. We shall show that one can replace the electromagnetic field vector, $A_\mu(x)$, by two scalar fields, $\Lambda(x)$ and $\Lambda'(x)$, together with a restricted vector field $\mathcal{A}_\mu(x)$, in such a way that the supplementary condition involves only the scalar fields, while the equation of motion for $\Psi[\sigma]$ contains only $\mathcal{A}_\mu(x)$, the sole physically significant part of the field. The decomposition will be conveniently expressed with the aid of an arbitrary time-like unit vector $n_\mu$; $n_\mu^2=-1$. The procedure of the customary theory corresponds to the special choice: $n_\mu=(0,0,0,i)$.

We decompose $A_\mu(x)$ into the gradient in the time-like direction specified by $n_\mu$ of a scalar operator $\Lambda(x)$, the gradient in the space-like direction orthogonal to $n_\mu$ of a scalar operator $\Lambda'(x)$ and the vector $\mathcal{A}_\mu(x)$ which has no component in the direction $n_\mu$, and is divergence-less. Symbolically

$$A_\mu(x)=n_\mu n_\nu\frac{\partial}{\partial x_\nu}\Lambda(x) -\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)\Lambda'(x)+\mathcal{A}_\mu(x), \quad (3.1)$$

where

$$n_\mu\mathcal{A}_\mu(x)=0, \quad \frac{\partial}{\partial x_\mu}\mathcal{A}_\mu(x)=0. \quad (3.2)$$

It is our first task to construct the commutation relations of the three fields thus defined, and in particular, to prove that they are kinematically independent. It follows from the definitions that

$$n_\mu A_\mu(x)=-n_\mu\frac{\partial}{\partial x_\mu}\Lambda(x) \quad (3.3)$$

and

$$\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)A_\mu(x)=-\left(n_\mu\frac{\partial}{\partial x_\mu}\right)^2\Lambda'(x), \quad (3.4)$$

in which the latter statement also involves the field equations

$$\Box^2\Lambda=\Box^2\Lambda'=\Box^2\mathcal{A}_\mu=0. \quad (3.5)$$

The commutation laws for $A_\mu(x)$ imply that

$$\left[n_\mu\frac{\partial}{\partial x_\mu}\Lambda(x),n_\nu\frac{\partial}{\partial x_\nu'}\Lambda(x')\right]=-i\hbar cD(x-x') \quad (3.6)$$

and

$$\left[\left(n_\mu\frac{\partial}{\partial x_\mu}\right)^2\Lambda'(x),\left(n_\nu\frac{\partial}{\partial x_\nu'}\right)^2\Lambda'(x')\right]=-i\hbar c\left(n_\mu\frac{\partial}{\partial x_\mu}\right)^2D(x-x'). \quad (3.7)$$

These can be simplified by the introduction of an odd function of the coordinates, $\mathfrak{D}(x)$, defined by

$$\left(n_\mu\frac{\partial}{\partial x_\mu}\right)^2\mathfrak{D}(x)=D(x)$$
$$\Box^2\mathfrak{D}(x)=0. \quad (3.8)$$

The relations (3.6) and (3.7) are satisfied if

$$[\Lambda(x),\Lambda(x')]=i\hbar c\mathfrak{D}(x-x')$$
$$[\Lambda'(x),\Lambda'(x')]=-i\hbar c\mathfrak{D}(x-x'). \quad (3.9)$$

The further commutation relation

$$[\Lambda(x),\Lambda'(x')]=0 \quad (3.10)$$

is consistent with the result deduced from (3.3) and (3.4), namely

$$\left[n_\mu\frac{\partial}{\partial x_\mu}\Lambda(x),\left(n_\nu\frac{\partial}{\partial x_\nu'}\right)^2\Lambda'(x')\right]=i\hbar c n_\mu\left(\frac{\partial}{\partial x_\mu'}+n_\mu n_\nu\frac{\partial}{\partial x_\nu'}\right)D(x-x')=0. \quad (3.11)$$

The commutation properties

$$[A_\mu(x)+n_\mu n_\nu A_\nu(x),n_\lambda A_\lambda(x')]=0,$$

$$\left[A_\mu(x)+\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\right)\Lambda'(x),\left(\frac{\partial}{\partial x_\nu'}+n_\nu n_\sigma\frac{\partial}{\partial x_\sigma'}\right)A_\nu(x')\right]=0, \quad (3.12)$$

when combined with the commutativity of $\Lambda$ and $\Lambda'$, imply that

$$[\Lambda(x),\mathfrak{A}_\mu(x')]=[\Lambda'(x),\mathfrak{A}_\mu(x')]=0. \quad (3.13)$$

Finally, we deduce from (3.1) that

$$[\mathfrak{A}_\mu(x),\mathfrak{A}_\nu(x')]=i\hbar c\delta_{\mu\nu}D(x-x')+i\hbar c n_\mu n_\nu\left(n_\lambda\frac{\partial}{\partial x_\lambda}\right)^2\mathfrak{D}(x-x')-i\hbar c\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\lambda\frac{\partial}{\partial x_\lambda}\right)\times\left(\frac{\partial}{\partial x_\nu}+n_\nu n_\sigma\frac{\partial}{\partial x_\sigma}\right)\mathfrak{D}(x-x') \quad (3.14)$$

or

$$[\mathfrak{A}_\mu(x),\mathfrak{A}_\nu(x')]=i\hbar c\delta_{\mu\nu}D(x-x')-i\hbar c\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}+\left(n_\mu\frac{\partial}{\partial x_\nu}+n_\nu\frac{\partial}{\partial x_\mu}\right)n_\lambda\frac{\partial}{\partial x_\lambda}\right)\mathfrak{D}(x-x'). \quad (3.15)$$

It will be noted that this commutation rule is compatible with the restrictions (3.2) imposed on $\mathfrak{A}_\mu(x)$.

The supplementary condition involves only the scalar fields and, indeed, only the combination $\Lambda(x)-\Lambda'(x)$, since

$$\frac{\partial}{\partial x_\mu}A_\mu(x)=\left(n_\mu\frac{\partial}{\partial x_\mu}\right)^2(\Lambda(x)-\Lambda'(x)). \quad (3.16)$$

Equation (2.32) will now be satisfied if

$$\left(\Lambda(x)-\Lambda'(x)-\int_\sigma\mathfrak{D}(x-x')\times\frac{1}{c}j_\mu(x')d\sigma_\mu'\right)\Psi[\sigma]=0. \quad (3.17)$$

The equation of motion for $\Psi[\sigma]$ becomes

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left\{-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)+\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}j_\mu(x)\Lambda'(x)\right)-\frac{1}{c}n_\mu j_\mu(x)n_\nu\frac{\partial}{\partial x_\nu}(\Lambda(x)-\Lambda'(x))\right\}\Psi[\sigma]. \quad (3.18)$$

It may be expected that, by a suitable gauge transformation, $\Lambda'(x)$ can be eliminated, leaving $\Lambda(x)-\Lambda'(x)$ and $\mathfrak{A}_\mu(x)$ as the fundamental variables of the electromagnetic field. Accordingly, we introduce the transformation (Cf. (2.40) and (2.41)).

$$\Psi[\sigma]\rightarrow e^{-iG'[\sigma]}\Psi[\sigma], \tag{3.19}$$

where

$$G'[\sigma]=\frac{1}{\hbar c}\int_\sigma \frac{1}{c}j_\mu(x)\Lambda'(x)d\sigma_\mu. \tag{3.20}$$

The new equations of motion and supplementary condition are:

$$\begin{aligned} ihc\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}&+ihce^{iG'[\sigma]}\frac{\delta e^{-iG'[\sigma]}}{\delta\sigma(x)}\Psi[\sigma] \\ &=e^{iG'[\sigma]}\bigg\{-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)+\frac{\partial}{\partial x_\mu}\bigg(\frac{1}{c}j_\mu(x)\Lambda'(x)\bigg) \\ &\quad-\frac{1}{c}n_\mu j_\mu(x)n_\nu\frac{\partial}{\partial x_\nu}(\Lambda(x)-\Lambda'(x))\bigg\} \\ &\qquad\times e^{-iG'[\sigma]}\Psi[\sigma] \end{aligned} \tag{3.21}$$

and

$$\bigg(\Lambda(x)-e^{iG'[\sigma]}\Lambda'(x)e^{-iG'[\sigma]} -\int_\sigma \mathfrak{D}(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu'\bigg)\Psi[\sigma]=0. \tag{3.22}$$

The transformation now under discussion differs from the previous gauge transformation in that $\Lambda'(x)$ is an operator, subject to the commutation relation (3.9). To indicate the modifications thereby introduced, we first evaluate

$$e^{iG'[\sigma]}\Lambda'(x)e^{-iG'[\sigma]}=\Lambda'(x)+i[G'[\sigma],\Lambda'(x)] -\frac{1}{2!}[G'[\sigma],[G'[\sigma],\Lambda'(x)]]+\cdots. \tag{3.23}$$

Now

$$\begin{aligned} i[G'[\sigma],\Lambda'(x)] &=-\frac{i}{\hbar c}\int_\sigma[\Lambda'(x),\Lambda'(x')]\frac{1}{c}j_\mu(x')d\sigma_\mu' \\ &=-\int_\sigma \mathfrak{D}(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu', \end{aligned} \tag{3.24}$$

and therefore, only the first two terms of the series (3.23) survive:

$$e^{iG'[\sigma]}\Lambda'(x)e^{-iG'[\sigma]} =\Lambda'(x)-\int_\sigma \mathfrak{D}(x-x')\frac{1}{c}j_\mu(x')d\sigma_\mu'. \tag{3.25}$$

Hence, the supplementary condition becomes, simply,

$$(\Lambda(x)-\Lambda'(x))\Psi[\sigma]=0. \tag{3.26}$$

In a similar way

$$\begin{aligned} e^{iG'[\sigma]}\bigg(\frac{\partial\Lambda'(x)}{\partial x_\mu}+n_\mu n_\nu\frac{\partial\Lambda'(x)}{\partial x_\nu}\bigg)e^{-iG'[\sigma]}&=\frac{\partial\Lambda'(x)}{\partial x_\mu} \\ +n_\mu n_\nu\frac{\partial\Lambda'(x)}{\partial x_\nu}-\int_\sigma\bigg(\frac{\partial}{\partial x_\mu}&+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\bigg) \\ &\times\mathfrak{D}(x-x')\frac{1}{c}j_\lambda(x'')d\sigma_\lambda', \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} ihce^{iG'[\sigma]}\frac{\delta e^{-iG'[\sigma]}}{\delta\sigma(x)}&=\frac{\partial}{\partial x_\mu}\bigg(\frac{1}{c}j_\mu(x)\Lambda'(x)\bigg) \\ &-\frac{1}{2}\int_\sigma\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu}\frac{1}{c}j_\mu(x)\frac{1}{c}j_\nu(x')d\sigma_\nu'. \end{aligned} \tag{3.28}$$

The equation of motion for $\Psi[\sigma]$ now reads

$$\begin{aligned} ihc\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg\{&-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)-\int_\sigma\bigg(\frac{1}{2}\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu} \\ &+n_\mu n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\bigg)\frac{1}{c}j_\mu(x)\frac{1}{c}j_\lambda(x')d\sigma_\lambda' \\ &-\frac{1}{c}n_\mu j_\mu(x)n_\nu\frac{\partial}{\partial x_\nu}(\Lambda(x)-\Lambda'(x))\bigg\}\Psi[\sigma] \end{aligned} \tag{3.29}$$

which, in view of the supplementary condition (3.26), reduces to

$$\begin{aligned} ihc\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\bigg\{&-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x) \\ &-\int_\sigma\bigg(\frac{1}{2}\frac{\partial\mathfrak{D}(x-x')}{\partial x_\mu}+n_\mu n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\bigg) \\ &\times\frac{1}{c}j_\mu(x)\frac{1}{c}j_\lambda(x')d\sigma_\lambda'\bigg\}\Psi[\sigma]. \end{aligned} \tag{3.30}$$

We have thereby succeeded in constructing an equation of motion for $\Psi[\sigma]$ which no longer contains the electromagnetic field variables involved in the supplementary condition. The additional term thus introduced is evidently the covariant generalization of the Coulomb interaction between charges.

To exhibit the latter property somewhat more clearly, we must restrict the arbitrary space-like surface $\sigma$ to a plane surface with the normal $n_\mu$. The advantage thereby acquired is the possibility of asserting that

$$\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)\mathfrak{D}(x-x')=0,\qquad n_\mu(x_\mu-x_\mu')=0 \tag{3.31}$$

which enables (3.30) to be simplified, yielding:

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left\{-\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)-\frac{1}{2}\int_\sigma n_\nu\frac{\partial\mathfrak{D}(x-x')}{\partial x_\nu}\times\frac{1}{c}n_\mu j_\mu(x)\frac{1}{c}j_\lambda(x')d\sigma_\lambda'\right\}\Psi[\sigma]. \tag{3.32}$$

To prove (3.31), it is sufficient to verify it in a particular coordinate system. It is always possible to construct a reference system for which the normal to a plane space-like surface is directed along the time axis; in other words, in this reference system, $n_\mu=(0,0,0,i)$. Equation (3.31) then states that the spatial derivatives of $\mathfrak{D}(\mathbf{r}-\mathbf{r}',t-t')$ vanish for $t=t'$. This will be true if $\mathfrak{D}(\mathbf{r},0)=0$ for all $\mathbf{r}$, that is, if $\mathfrak{D}(\mathbf{r},t)$ is an odd function of $t$. Now, in this special coordinate system, (3.8) becomes

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2\mathfrak{D}(\mathbf{r},t)=\nabla^2\mathfrak{D}(\mathbf{r},t)=D(\mathbf{r},t) \tag{3.33}$$

and, since (2.17) assures us that $D(\mathbf{r},t)$ is an odd function of the time, the necessary property of $\mathfrak{D}(\mathbf{r},t)$ is established. As a final step, we note that, in this special coördinate system,

$$n_\nu\frac{\partial}{\partial x_\nu}\mathfrak{D}(x-x')=-\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{D}(\mathbf{r}-\mathbf{r}',0),\qquad n_\mu(x_\mu-x_\mu')=0 \tag{3.34}$$

and

$$\nabla^2\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{D}(\mathbf{r},0)=\frac{1}{c}\frac{\partial}{\partial t}D(\mathbf{r},0)=-\delta(\mathbf{r}), \tag{3.35}$$

in which the latter statement involves the content of the equations of definition (2.17) as adapted to the special coordinate system. It follows from (3.35) that

$$-\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{D}(\mathbf{r},0)=\frac{1}{4\pi r}, \tag{3.36}$$

of which the covariant expression is

$$n_\nu\frac{\partial}{\partial x_\nu}\mathfrak{D}(x-x')=\frac{1}{4\pi}\frac{1}{[(x_\mu-x_\mu')^2]^{\frac{1}{2}}},\qquad n_\mu(x_\mu-x_\mu')=0. \tag{3.37}$$

The energy-momentum four-vector is modified by the unitary transformation (3.19), according to

$$P_\nu[\sigma]\rightarrow e^{iG'[\sigma]}P_\nu[\sigma]e^{-iG'[\sigma]}. \tag{3.38}$$

The evaluation of the new operator $P_\nu[\sigma]$ involves the following transformations, which we note without proof:

$$e^{iG'[\sigma]}A_\mu(x)e^{-iG'[\sigma]}=A_\mu(x)+\int_\sigma\left(\frac{\partial}{\partial x_\mu}+n_\mu n_\nu\frac{\partial}{\partial x_\nu}\right)\mathfrak{D}(x-x')\frac{1}{c}j_\lambda(x')d\sigma_\lambda',$$

$$e^{iG'[\sigma]}\frac{1}{2}\int_\sigma d\sigma_\mu\left[\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\mu\psi\right]e^{-iG'[\sigma]}$$
$$=\frac{1}{2}\int_\sigma d\sigma_\mu\left[\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\mu\psi\right]+\frac{1}{\hbar c}\int_\sigma\left[d\sigma_\nu\frac{1}{c}j_\lambda\frac{\partial\Lambda'}{\partial x_\lambda}-d\sigma_\lambda\frac{1}{c}j_\lambda\frac{\partial\Lambda'}{\partial x_\nu}\right]. \tag{3.39}$$

In virtue of the supplementary condition (3.26), we may write

$$A_\mu(x)\Psi[\sigma]=\left(\mathfrak{A}_\mu(x)-\frac{\partial\Lambda'(x)}{\partial x_\mu}\right)\Psi[\sigma]. \tag{3.40}$$

It follows, as a result of straightforward simplification, that

$$P_\nu[\sigma]=\frac{1}{2c}\int_\sigma d\sigma_\mu\left[\frac{\partial\mathfrak{A}_\lambda}{\partial x_\mu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\mu}-\delta_{\mu\nu}\left(\frac{\partial\mathfrak{A}_\lambda}{\partial x_\sigma}\right)^2\right]+\frac{\hbar}{2}\int_\sigma d\sigma_\mu\left[\bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\bar\psi}{\partial x_\nu}\gamma_\mu\right]+\frac{1}{c}\int_\sigma d\sigma_\nu\left[\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)+\frac{1}{2}\int_\sigma n_\zeta\frac{\partial\mathfrak{D}(x-x')}{\partial x_\zeta}\times\frac{1}{c}n_\mu j_\mu(x)\frac{1}{c}j_\lambda(x')d\sigma_\lambda'\right],\quad(3.41)$$

which has been stated as an operator equation, rather than a derived supplementary condition, with the understanding that the operator $\Lambda-\Lambda'$ shall no longer appear in the theory.

As the final comment of this section, we remark that the derivatives of $\mathfrak{A}_\lambda$, occurring in (3.41), can be combined into the field strengths

$$\mathfrak{F}_{\mu\nu}=\frac{\partial}{\partial x_\mu}\mathfrak{A}_\nu-\frac{\partial}{\partial x_\nu}\mathfrak{A}_\mu\qquad(3.42)$$

with the result that

$$P_\nu[\sigma]=\frac{1}{2c}\int_\sigma d\sigma_\mu[\mathfrak{F}_{\mu\lambda}\mathfrak{F}_{\nu\lambda}+\mathfrak{F}_{\nu\lambda}\mathfrak{F}_{\mu\lambda}-\tfrac{1}{2}\delta_{\mu\nu}\mathfrak{F}_{\lambda\sigma}{}^2]=\frac{\hbar}{2}\int_\sigma d\sigma_\mu\left[\bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\bar\psi}{\partial x_\nu}\gamma_\mu\right]+\frac{1}{c}\int_\sigma d\sigma_\nu\left[\frac{1}{c}j_\mu(x)\,\mathfrak{A}_\mu(x)+\frac{1}{2}\int_\sigma n_\zeta\frac{\partial\mathfrak{D}(x-x')}{\partial x_\zeta}\times\frac{1}{c}n_\mu j_\mu(x)\frac{1}{c}j_\lambda(x')d\sigma_\lambda'\right].\quad(3.43)$$

We need only notice that (Cf. (1.32))

$$\frac{\partial\mathfrak{A}_\lambda}{\partial x_\mu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\mu}-\delta_{\mu\nu}\left(\frac{\partial\mathfrak{A}_\lambda}{\partial x_\sigma}\right)^2=\mathfrak{F}_{\mu\lambda}\mathfrak{F}_{\nu\lambda}+\mathfrak{F}_{\nu\lambda}\mathfrak{F}_{\mu\lambda}-\tfrac{1}{2}\delta_{\mu\nu}\mathfrak{F}_{\lambda\sigma}{}^2+\frac{\partial}{\partial x_\lambda}(\mathfrak{A}_\nu\mathfrak{F}_{\mu\lambda}+\mathfrak{F}_{\mu\lambda}\mathfrak{A}_\nu)+\frac{\partial\mathfrak{A}_\mu}{\partial x_\lambda}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}\frac{\partial\mathfrak{A}_\mu}{\partial x_\lambda}-\delta_{\mu\nu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\sigma}\frac{\partial\mathfrak{A}_\sigma}{\partial x_\lambda}.\quad(3.44)$$

The lemma (1.58), together with the antisymmetry of $\mathfrak{F}_{\mu\lambda}$, then informs us that

$$\int_\sigma d\sigma_\mu\frac{\partial}{\partial x_\lambda}(\mathfrak{A}_\nu\mathfrak{F}_{\mu\lambda}+\mathfrak{F}_{\mu\lambda}\mathfrak{A}_\nu)=0,$$

while the divergence-less nature of $\mathfrak{A}_\mu$ is combined with (1.58) in the following proof:

$$\int_\sigma d\sigma_\mu\left[\frac{\partial\mathfrak{A}_\mu}{\partial x_\lambda}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}\frac{\partial\mathfrak{A}_\mu}{\partial x_\lambda}-\delta_{\mu\nu}\frac{\partial\mathfrak{A}_\lambda}{\partial x_\sigma}\frac{\partial\mathfrak{A}_\sigma}{\partial x_\lambda}\right]=\frac{1}{2}\int_\lambda d\sigma_\mu\frac{\partial}{\partial x_\lambda}\left[\mathfrak{A}_\mu\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\lambda}{\partial x_\nu}\mathfrak{A}_\mu+\mathfrak{A}_\lambda\frac{\partial\mathfrak{A}_\mu}{\partial x_\nu}+\frac{\partial\mathfrak{A}_\mu}{\partial x_\nu}\mathfrak{A}_\lambda\right]-\int_\sigma d\sigma_\nu\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\sigma}(\mathfrak{A}_\lambda\mathfrak{A}_\sigma)=\frac{1}{2}\int_\sigma d\sigma_\mu\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\lambda}(\mathfrak{A}_\mu\mathfrak{A}_\lambda+\mathfrak{A}_\lambda\mathfrak{A}_\mu)-\int_\sigma d\sigma_\nu\frac{\partial}{\partial x_\lambda}\frac{\partial}{\partial x_\sigma}(\mathfrak{A}_\lambda\mathfrak{A}_\sigma)=0.\quad(3.45)$$

### 4. THE INVARIANT COLLISION OPERATOR

While the interactions between fields and their vacuum fluctuations are conveniently regarded as modifying the properties of the non-interacting fields, other types of interactions are often best viewed as producing transitions among the states of the individual fields. We shall conclude this paper with a brief discussion of a covariant manner of describing such transitions. The change in state of several fields arising from their mutual interaction is described by the equation of motion (2.6) for the state vector $\Psi[\sigma]$. The question that must be answered in order to describe collisions between the particles associated with the quantized fields is: given the state vector on a surface $\sigma_1$, what is the state vector on the surface $\sigma_2$, in the limit as $\sigma_1$ and $\sigma_2$ recede into the remote and past and future, respectively? In this limit, no precise characterization of the surfaces is required and we shall accordingly denote them by the symbols $-\infty$ and $+\infty$, respectively. It will be useful to derive the state vector on an arbitrary surface $\sigma$ from that for the initial surface $\sigma_1$ by a unitary operator:

$$\Psi[\sigma]=U[\sigma,\sigma_1]\Psi[\sigma_1]\qquad(4.1)$$

which is to be determined by an equation of motion

$$i\hbar c\frac{\delta}{\delta\sigma(x)}U[\sigma,\sigma_1]=\mathcal{H}(x)U[\sigma,\sigma_1], \quad (4.2)$$

and an initial condition

$$U[\sigma_1,\sigma_1]=1. \quad (4.3)$$

The functional differential equation (4.2) is conveniently replaced by a functional integral equation, which incorporates the initial condition (4.3):

$$U[\sigma,\sigma_1]=1-\frac{i}{\hbar c}\int_{\sigma_1}^{\sigma}\mathcal{H}(x')U[\sigma',\sigma_1]d\omega'. \quad (4.4)$$

The volume integral in this equation is extended between the surfaces $\sigma_1$ and $\sigma$. In particular,

$$U[\sigma_2,\sigma_1]=1-\frac{i}{\hbar c}\int_{\sigma_1}^{\sigma_2}\mathcal{H}(x)U[\sigma,\sigma_1]d\omega. \quad (4.5)$$

In terms of the limiting surfaces $\pm\infty$, the integral equation becomes

$$U[\sigma,-\infty]=1-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}\mathcal{H}(x')U[\sigma',-\infty]d\omega', \quad (4.6)$$

while

$$S=1-\frac{i}{\hbar c}\int_{-\infty}^{\infty}\mathcal{H}(x)U[\sigma,-\infty]d\omega. \quad (4.7)$$

Here

$$S=U[\infty,-\infty], \quad (4.8)$$

which we call the collision operator, determines the over-all change in state of the system as the result of interaction:

$$\Psi[\infty]=S\Psi[-\infty]. \quad (4.9)$$

The expectation value of some physical quantity $F$ can then be calculated in the final state, for a prescribed initial state:

$$(\Psi[\infty],F\Psi[\infty]) =(\Psi[-\infty],S^{-1}FS\Psi[-\infty]) \quad (4.10)$$

from which the probabilities of various transitions can be inferred.

The problem of determining the unitary collision operator $S$ can be replaced by that of determining a Hermitian operator $K$, which we may call the reaction operator. Our procedure will be precisely analogous to the use, in classical electrodynamics, of the sum and difference of advanced and retarded potentials rather than the latter alone. The integral equation (4.6) can be rewritten as

$$\begin{aligned}U[\sigma,-\infty]&+\frac{i}{2\hbar c}\left[\int_{-\infty}^{\sigma}\mathcal{H}(x')U[\sigma',-\infty]d\omega' -\int_{\sigma}^{\infty}\mathcal{H}(x')U[\sigma',-\infty]d\omega'\right]\\ &=1-\frac{i}{2\hbar c}\left[\int_{-\infty}^{\sigma}\mathcal{H}(x')U[\sigma',-\infty]d\omega' +\int_{\sigma}^{\infty}\mathcal{H}(x')U[\sigma',-\infty]d\omega'\right], \quad (4.11)\end{aligned}$$

or

$$\begin{aligned}U[\sigma,-\infty]&+\frac{i}{2\hbar c}\int_{-\infty}^{\infty}\epsilon[\sigma,\sigma']\mathcal{H}(x')U[\sigma',-\infty]d\omega'\\ &=1-\frac{i}{2\hbar c}\int_{-\infty}^{\infty}\mathcal{H}(x')U[\sigma',-\infty]d\omega', \quad (4.12)\end{aligned}$$

where $\epsilon[\sigma,\sigma']=1$ if $\sigma'$ antedates the surface $\sigma$ while $\epsilon[\sigma,\sigma']=-1$ if $\sigma'$ succeeds the surface $\sigma$. It is useful to introduce the functional $V[\sigma]$ according to the definition

$$\begin{aligned}U[\sigma,-\infty]&=V[\sigma]\left(1-\frac{i}{2\hbar c}\int_{-\infty}^{\infty}\mathcal{H}(x')U[\sigma',-\infty]d\omega'\right)\\ &=V[\sigma]\tfrac{1}{2}(1+S) \quad (4.13)\end{aligned}$$

so that

$$V[\sigma]+\frac{i}{2\hbar c}\int_{-\infty}^{\infty}\epsilon[\sigma,\sigma']\mathcal{H}(x')V[\sigma']d\omega'=1. \quad (4.14)$$

On computing $S$ from (4.7) and (4.13), we learn that

$$i\frac{S-1}{S+1}=\frac{1}{2\hbar c}\int_{-\infty}^{\infty}\mathcal{H}(x)V[\sigma]d\omega=K, \quad (4.15)$$

or

$$S=\frac{1-iK}{1+iK}. \quad (4.16)$$

While the Hermitian character of $K$ is an immediate consequence of the unitary nature of $S$, it is instructive to give a direct proof. Associated with the integral equation for $V[\sigma]$ is the Hermitian conjugate equation

$$V^+[\sigma]+\frac{i}{2\hbar c}\int_{-\infty}^{\infty} d\omega' \, V^+[\sigma']\mathcal{H}(x')\epsilon[\sigma',\sigma]=1, \quad (4.17)$$

in which we have used the evident relation

$$\epsilon[\sigma,\sigma'] = -\epsilon[\sigma',\sigma] \qquad (4.18)$$

and the Hermiticity of $\mathcal{H}(x)$. We may now multiply (4.14) to the left with $V^+[\sigma]\mathcal{H}(x)$, multiply (4.17) to the right with $\mathcal{H}(x)V[\sigma]$, and integrate with respect to $x$ over all space-time. A comparison of the resultant formulae yields:

$$\int_{-\infty}^{\infty} V^+[\sigma]\mathcal{H}(x)V[\sigma]d\omega+\frac{i}{2\hbar c} \times\int_{-\infty}^{\infty} V^+[\sigma]\mathcal{H}(x)\epsilon[\sigma,\sigma']\mathcal{H}(x')V[\sigma']d\omega d\omega' =\int_{-\infty}^{\infty}\mathcal{H}(x)V[\sigma]d\omega=\int_{-\infty}^{\infty}V^+[\sigma]\mathcal{H}(x)d\omega \quad (4.19)$$

or

$$K=K^+. \qquad (4.20)$$

An important stationary property of the reaction operator $K$ should be noted. On writing (4.19) as

$$2\hbar c\int_{-\infty}^{\infty} V^+[\sigma]\mathcal{H}(x)V[\sigma]d\omega +i\int_{-\infty}^{\infty} V^+[\sigma]\mathcal{H}(x)\epsilon[\sigma,\sigma']\mathcal{H}(x')V[\sigma']d\omega d\omega' =\int_{-\infty}^{\infty}V^+[\sigma]\mathcal{H}(x)d\omega K^{-1} \times\int_{-\infty}^{\infty}\mathcal{H}(x')V[\sigma']d\omega', \quad (4.21)$$

we obtain a formula for $K$ which is homogeneous in $V[\sigma]$ and $V^+[\sigma]$ and stationary with respect to small variations of $V[\sigma]$ and $V^+[\sigma]$. On performing such a variation, we obtain

$$-\int_{-\infty}^{\infty}V^+[\sigma]\mathcal{H}(x)d\omega K^{-1}\delta K K^{-1}\int_{-\infty}^{\infty}\mathcal{H}(x)V[\sigma]d\omega=2\hbar c\int_{-\infty}^{\infty}d\omega\,\delta V^+[\sigma]\mathcal{H}(x)\bigg[V^+[\sigma] +\frac{i}{2\hbar c}\int_{-\infty}^{\infty}\epsilon[\sigma,\sigma']\mathcal{H}(x')V[\sigma']d\omega'-\frac{1}{2\hbar c}K^{-1}\int_{-\infty}^{\infty}\mathcal{H}(x')V[\sigma']d\omega'\bigg]+2\hbar c\int_{-\infty}^{\infty}\bigg[V^+[\sigma]+\frac{i}{2\hbar c} \times\int_{-\infty}^{\infty}V^+[\sigma']\mathcal{H}(x')\epsilon[\sigma',\sigma]d\omega'-\frac{1}{2\hbar c}\int_{-\infty}^{\infty}V^+[\sigma']\mathcal{H}(x')d\omega' K^{-1}\bigg]\mathcal{H}(x)\,\delta V[\sigma]d\omega. \quad (4.22)$$

Evidently, if $V[\sigma]$ satisfies (4.14) and (4.15), together with the Hermitian adjoint equations for $V^+[\sigma]$, $\delta K=0$. Conversely, if $K$ is stationary for arbitrary variations, the quantities within brackets on the right side of (4.22) must vanish. It is easily seen that the functional

$$V'[\sigma]=V[\sigma]\left(\int_{-\infty}^{\infty}\mathcal{H}(x')V[\sigma']d\omega'\right)^{-1}2\hbar cK \quad (4.23)$$

obeys Eqs. (4.14) and (4.15), while $V'^+[\sigma]$ obey the corresponding Hermitian adjoint equations. This type of variational principle has been extensively applied in the treatment of scattering problems,[12] and will be discussed in detail elsewhere.

As a final remark, we observe that the representation of $S$ as an integral extended over all space-time indicates that it is unaffected by a translation of the coordinate system, and therefore commutes with the operator $P_\mu^{(0)}$ (cf. Eq. (2.50)):

$$[S,P_\mu^{(0)}]=0. \qquad (4.24)$$

This is the energy-momentum conservation law for collision processes, since, according to (4.10), the expectation value of $P_\mu^{(0)}$ is unchanged by the course of interaction, for an arbitrary initial state.

[12] J. Schwinger, Phys. Rev. 72, 742 (1947) and unpublished lecture notes.